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Normal frames for general connections on differentiable fibre bundles

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Abstract

The theory of frames normal for general connections on differentiable bundles is developed. Links with the existing theory of frames normal for covariant derivative operators (linear connections) in vector bundles are revealed. The existence of bundle coordinates normal at a given point and/or along injective horizontal path is proved. A necessary and sufficient condition of existence of bundle coordinates normal along injective horizontal mappings is derived.

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1. Introduction

In general, frames and coordinates in which some geometrical object has vanishing components are called "normal"; in particular, these can be the coefficients of a linear connection on a manifold or in vector bundle. As a result of that, some objects look in them like in a "flat" or "Euclidean" case, which significantly simplifies certain calculations, formulae, their interpretation, etc. For instance, the normal frames for linear connections turn to be the mathematical object for description of the inertial frames of reference in physics, in which some effects of a force field, like the gravity one, locally disappear.

The history of the theory of normal coordinates and frames goes back to 1854. The major classical results concerning the normal coordinates for linear connections are summarize in the table below.

Year	Person	Result and original reference
1854	B. Riemann	Existence and construction of ('Riemannian') coordinates in a Riemannian manifold which are normal at a single point [1]
1922	O. Veblen	Existence and construction of ('Riemannian normal') coordinates in a manifold with torsionless linear connection which are normal at a single point [2]
1922	E. Fermi	Existence of ('Fermi') coordinates in a Riemannian manifold which are normal along a path without self-intersections [3]
1926	T. Levi-Civita	Explicit transformation to the Fermi coordinates along paths without self-intersections [4]
1927	L.P. Eisenhart	Existence and construction of particular kind of ('Fermi') coordinates on a manifold with torsionless linear connection which are normal along a path without self-intersections [5]
1958	L. O'Raifeartaigh	Necessary and sufficient conditions for existence of coordinates normal on submanifold of a manifold with torsionless linear connection. If such coordinates exist, a particular example of them ('Fermi coordinates') is constructed [6]

In [7–9] the normal frames were introduced and studied for derivations, in particular for linear connections, with generally non-vanishing curvature and torsion on a differentiable manifold. Then these objects were investigated for derivations and linear connections in vector bundles [10]. At last, the paper [11] explores them for linear transports along paths in vector bundles. The present work is devoted to the introduction and some properties of normal frames and coordinates for general connections on fibre bundles whose bundle and base spaces are differentiable manifolds.

The layout of the work is as follows. In Section 2 is collected some introductory material needed for our exposition. Here some of our notation is fixed too.

Section 3 is devoted to the general connection theory on bundles whose base and bundles spaces are differentiable manifolds. In Section 3.1 are reviewed some coordinates and frames/bases on the bundle space which are compatible with the fibre structure of a bundle. Section 3.2 deals with the general connection theory. A connection on a bundle is defined as a distribution on its bundle space which is complimentary to the vertical distribution on it. The notion of specialized frame is introduced. Frames adapted to specialized frames, in particular to local bundle coordinates, are defined and the local (2-index) coefficients in them of a connection are defined and their transformation law is derived.

The theory of normal frames for connections on bundles is considered in Sections 4–6. Section 4 deals with the general case. Loosely said, an adapted frame is called normal if the 2-index coefficients of a connection vanish in it (on some set). It happens that a frame is normal if and only if it coincides with the frame it is adapted to. The set of these frames is completely described in the most general case. The problems of existence, uniqueness, etc. of normal frames adapted to holonomic frames, i.e. adapted to local coordinates, are discussed in Section 5. If such frames exist, their general form is described. The existence of frames normal at a given point and/or along an injective horizontal path is proved. The flatness of a connection on an open set is pointed as a necessary condition of existence of (locally) holonomic frames normal on that set. Some links between the general theory of normal frames and the existing one of normal frames in vector bundles are given in Section 6. It is proved that a frame is normal on a vector bundle with linear connection if and only if in it vanish the 3-index coefficients of the connection. The equivalence of the both theories on vector bundles is established.

Section 7 ends the paper with some concluding remarks.

In Appendix A is formulated and proved a necessary and sufficient condition for the existence of coordinates normal along injective mappings with non-vanishing horizontal component, in particular along injective horizontal mappings.

2. Preliminaries

This section contains an introductory material, notation, etc. that will be needed for our exposition. The reader is referred for details to standard books on differential geometry, like [12–14].

A differentiable finite-dimensional manifold over a field $\mathbb{K}$ will be denoted typically by M. Here $\mathbb{K}$ stands for the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers, $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The manifolds we consider are supposed to be smooth of class C^2.[1] The set of vector fields, realized as first order differential operators, over M will be denoted by $\mathcal{X}(M)$. The space tangent to M at $p \in M$ is $T_p(M)$ and $(T(M), \pi_T, M)$ will stand for the tangent bundle over M. The value of $X \in \mathcal{X}(M)$ at $p \in M$ is $X_p \in T_p(M)$.

If M and $\bar{M}$ are manifolds and $f : \bar{M} \to M$ is a C^1 mapping, then $f_* := \mathrm{d}f : T(\bar{M}) \to T(M)$ denotes the induced tangent mapping (or differential) of f such that, for $p \in M$, $f_*|_p := \mathrm{d}f|_p : T_p(\bar{M}) \to T_{f(p)}(M)$ and, for a C^1 function g on M, $(f_*(X))(g) := X(g \circ f) : p \mapsto f_*|_p(g) = X_p(g \circ f)$, with $\circ$ being the composition of mappings sign.

By $J \subseteq \mathbb{R}$ will be denoted an arbitrary real interval that can be open or closed at one or both its ends. The notation $\gamma : J \to M$ represents an arbitrary path in M. For a C^1 path $\gamma : J \to M$, the vector tangent to γ at $s \in J$ will be denoted by $\dot{\gamma}(s) := (\mathrm{d}/\mathrm{d}t)|_{t=s}(\gamma(t)) = \gamma_*((\mathrm{d}/\mathrm{d}r)|_s) \in T_{\gamma(s)}(M)$, where r in $(\mathrm{d}/\mathrm{d}r)|_s$ is the standard coordinate function on $\mathbb{R}$, i.e. $r : \mathbb{R} \to \mathbb{R}$ with $r(s) := s$ for all $s \in \mathbb{R}$ and hence $r = \mathsf{id}_{\mathbb{R}}$ is the identity mapping of $\mathbb{R}$. If

[1] Some of our definitions or/and results are valid also for C^1 or even C^0 manifolds, but we do not want to overload the material with continuous counting of the required degree of differentiability of the manifolds involved. Some parts of the text admit generalizations on more general spaces, like the topological ones, but this is out of the subject of the present work.

$s_0 \in J$ is an end point of J and J is closed at s_0, the derivative in the definition of $\dot{\gamma}(s_0)$ is regarded as a one-sided derivative at s_0.

Let the Greek indices $\lambda, \mu, \nu, \dots$ run over the range $1, \dots, \dim M$ and $\{E_\mu\}$ be a C^1 frame in $T(M)$, i.e. $E_\mu \in \mathcal{X}(M)$ be of class C^1 and, for each $p \in M$, the set $\{E_\mu|_p\}$ to be a basis of the vector space $T_p(M)$. [2] The Einstein's summation convention, summation on indices repeated on different levels over the whole range of their values, will be assumed hereafter.

A frame $\{E_\mu\}$ or its dual coframe $\{E^\mu\}$ is called *holonomic* (*anholonomic*) if $C^\lambda_{\mu\nu} = 0$ ($C^\lambda_{\mu\nu} \neq 0$) for all (some) values of the indices μ, ν, and λ, where the functions $C^\nu_{\mu\lambda}$ are defined by $[E_\mu, E_\nu]_- := E_\mu \circ E_\nu - E_\nu \circ E_\mu =: C^\lambda_{\mu\nu} E_\lambda$; these functions are a measure of deviation from a holonomic frame and are known as the *components of the anholonomy object of* $\{E_\mu\}$. For a holonomic frame there always exist local coordinates $\{x^\mu\}$ on M such that *locally* $E_\mu = \partial/\partial x^\mu$ and $E^\mu = \mathrm{d}x^\mu$. Conversely, if $\{x^\mu\}$ are local coordinates on M, then the local frame $\{\partial/\partial x^\mu\}$ and local coframe $\{\mathrm{d}x^\mu\}$ are well defined and holonomic on the domain of $\{x^\mu\}$.

If $n \in \mathbb{N}$ and $n \leq \dim M$, an n-dimensional *distribution* Δ on M is defined as a mapping $\Delta : p \mapsto \Delta_p$ assigning to each $p \in M$ an n-dimensional subspace Δ_p of the tangent space $T_p(M)$ of M at p, $\Delta_p \subseteq T_p(M)$. A distribution is *integrable* if there is a submersion $\psi : M \to N$ such that $\operatorname{Ker} \psi_* = \Delta$; a necessary and locally sufficient condition for the integrability of Δ is the commutator of every two vector fields in Δ to be in Δ. We say that a vector field $X \in \mathcal{X}(M)$ is in Δ and write $X \in \Delta$, if $X_p \in \Delta_p$ for all $p \in M$. A *basis on* $U \subseteq M$ *for* Δ is a set $\{X_1, \dots, X_n\}$ of n linearly independent (relative to functions $U \to \mathbb{K}$) vector fields in $\Delta|_U$, i.e. $\{X_1|_p, \dots, X_n|_p\}$ is a basis for Δ_p for all $p \in U$.

A distribution is convenient to be described in terms of (global) frames or/and coframes over M. In fact, if $p \in M$ and $\varrho = 1, \dots, n$, in each $\Delta_p \subseteq T_p(M)$, we can choose a basis $\{X_\varrho|_p\}$ and hence a frame $\{X_\varrho\}$, $X_\varrho : p \mapsto X_\varrho|_p$, in $\{\Delta_p : p \in M\} \subseteq T(M)$; we say that $\{X_\rho\}$ is a basis for/in Δ. Conversely, any collection of n linearly independent (relative to functions $M \to \mathbb{K}$) vector fields X_ϱ on M defines a distribution $p \mapsto \{\sum_{\varrho=1}^n f^\varrho X_\varrho|_p : f^\varrho \in \mathbb{K}\}$. Consequently, a frame in $T(M)$ can be formed by adding to a basis for Δ a set of $(\dim M - n)$ new linearly independent vector fields (forming a frame in $T(M) \setminus \{\Delta_p : p \in M\}$) and v.v., by selecting n linearly independent vector fields on M, we can define a distribution Δ on M.

3. Connections on bundles

Before presenting the general connection theory in Section 3.2, we at first fix some notation and concepts concerning fibre bundles in Section 3.1.

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[2] There are manifolds, like the even-dimensional spheres $\mathbb{S}^{2k}$, $k \in \mathbb{N}$, which do not admit global, continuous (and moreover C^k for $k \geq 1$), and nowhere vanishing vector fields [15]. If this is the case, the considerations must be localized over an open subset of M on which such fields exist. We shall not overload our exposition with such details.

3.1. Coordinates and frames on the bundle space

Let (E, π, M) be a bundle with bundle space E, projection $\pi : E \to M$, and base space M. We suppose that the spaces M and E are C^2 differentiable, if the opposite is not stated explicitly,[3] manifolds of finite dimensions $n \in \mathbb{N}$ and $n + r$, for some $r \in \mathbb{N}$, respectively; so the dimension of the fibres $\pi^{-1}(x)$, with $x \in M$, i.e. the fibre dimensions of (E, π, M), is r.

Let the Greek indices $\lambda, \mu, \nu, \dots$ run from 1 to $n = \dim M$, the Latin indices $a, b, c, \dots$ take the values from $n + 1$ to $n + r = \dim E$, and the uppercase Latin indices $I, J, K, \dots$ take values in the whole set $\{1, \dots, n + r\}$. One may call these types of indices respectively base, fibre, and bundle indices.

Suppose $\{u^I\} = \{u^\mu, u^a\} = \{u^1, \dots, u^{n+r}\}$ are local bundle coordinates on an open set $U \subseteq E$, i.e. on the set $\pi(U) \subseteq M$ there are local coordinates $\{x^\mu\}$ such that $u^\mu = x^\mu \circ \pi$; the coordinates $\{u^\mu\}$ (resp. $\{u^a\}$) are called *basic* (resp. *fibre*) *coordinates* [14]. [4]

Further only coordinate changes

$$\{u^\mu, u^a\} \mapsto \{\tilde{u}^\mu, \tilde{u}^a\} \tag{3.1a}$$

on E between bundle coordinates will be considered. This means that

$$\begin{aligned} &\tilde{u}^\mu(p) = f^\mu(u^1(p), \dots, u^n(p)), \\ &\tilde{u}^a(p) = f^a(u^1(p), \dots, u^n(p), u^{n+1}(p), \dots, u^{n+r}(p)) \end{aligned} \tag{3.1b}$$

for $p \in E$ and some functions f^I. The bundle coordinates $\{u^\mu, u^a\}$ induce the (local) frame $\{\partial_\mu := \partial/\partial u^\mu, \partial_a := \partial/\partial u^a\}$ over U in the tangent bundle space $T(E)$ of the tangent bundle over the bundle space E. Since a change (3.1) of the coordinates on E implies $\partial_I \mapsto \tilde{\partial}_I := \partial/\partial\tilde{u}^I = \partial u^J/\partial\tilde{u}^I \partial_J$, the transformation (3.1) leads to

$$(\partial_\mu, \partial_a) \mapsto (\tilde{\partial}_\mu, \tilde{\partial}_a) = (\partial_\nu, \partial_b) \cdot A. \tag{3.2}$$

Here expressions like $(\partial_\mu, \partial_a)$ are shortcuts for ordered $(n + r)$-tuples like $(\partial_1, \dots, \partial_{n+r}) = ([\partial_\mu]_{\mu=1}^n, [\partial_a]_{a=n+1}^{n+r})$, the centered dot $\cdot$ stands for the matrix multiplication, and the transformation matrix A is

$$A := \left[\frac{\partial u^I}{\partial\tilde{u}^J}\right]_{I,J=1}^{n+r} = \begin{pmatrix} \left[\frac{\partial u^\nu}{\partial\tilde{u}^\mu}\right] & 0_{n\times r} \\ \left[\frac{\partial u^b}{\partial\tilde{u}^\mu}\right] & \left[\frac{\partial u^b}{\partial\tilde{u}^a}\right] \end{pmatrix} =: \begin{bmatrix} \frac{\partial u^\nu}{\partial\tilde{u}^\mu} & 0 \\ \frac{\partial u^b}{\partial\tilde{u}^\mu} & \frac{\partial u^b}{\partial\tilde{u}^a} \end{bmatrix}, \tag{3.3}$$

where $0_{n\times r}$ is the $n \times r$ zero matrix. Besides, in expressions of the form $\partial_I a^I$, like the one in the r.h.s. of (3.2), the summation excludes differentiation, i.e. $\partial_I a^I := a^I \partial_I = \sum_I a^I \partial_I$; if

[3] Most of our considerations are valid also if C^1 differentiability is assumed and even some of them hold on C^0 manifolds. By assuming C^2 differentiability, we skip the problem of counting the required differentiability class of the whole material that follows. Sometimes, the C^2 differentiability is required explicitly, which is a hint that a statement or definition is not valid otherwise. If we want to emphasize that some text is valid under a C^1 differentiability assumption, we indicate that fact explicitly. However, the proofs of Lemmas 5.1 and A.1, Proposition 5.3 and all assertions in Appendix A require C^3 differentiability, which will be indicated explicitly.

[4] If (U, v) is a bundle chart, with $v : U \to \mathbb{K}^n \times \mathbb{K}^r$ and $e^a : \mathbb{K}^r \to \mathbb{K}$ are such that $e^a(c_1, \dots, c_r) = c_a \in \mathbb{K}$, then one can put $u^a = e^a \circ \mathrm{pr}_2 \circ v$, where $\mathrm{pr}_2 : \mathbb{K}^n \times \mathbb{K}^r \to \mathbb{K}^r$ is the projection on the second multiplier $\mathbb{K}^r$.

a differentiation really takes place, we write $\partial_I(a^I) := \sum_I \partial_I(a^I)$. This rule allows a lot of formulae to be written in a compact matrix form, like (3.2). The explicit form of the matrix inverse to (3.3) is $A^{-1} = [\partial\tilde{u}^I/\partial u^J] = \cdots$ and it is obtained from (3.3) via the change $u \leftrightarrow \tilde{u}$.

The formula (3.2) can be generalized for arbitrary frames $\{e_I\} = \{e_\mu, e_a\}$ and $\{\tilde{e}_I\} = \{\tilde{e}_\mu, \tilde{e}_a\}$ in $T(E)$ whose *admissible changes* are given by

$$(e_I) = (e_\mu, e_a) \mapsto (\tilde{e}_I) = (\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot A, \tag{3.4}$$

where $A = [A^I_J]$ is a nondegenerate matrix-valued function with a block structure similar to (3.3), viz.

$$A = \begin{pmatrix} [A^\nu_\mu]^n_{\mu,\nu=1} & 0_{n\times r} \\ [A^b_\mu]_{\substack{\mu=1,\dots,n \\ b=n+1,\dots,n+r}} & [A^b_a]^{n+r}_{a,b=n+1} \end{pmatrix} =: \begin{bmatrix} A^\nu_\mu & 0 \\ A^b_\mu & A^b_a \end{bmatrix} \tag{3.5a}$$

with inverse matrix

$$A^{-1} = \begin{pmatrix} [A^\nu_\mu]^{-1} & 0 \\ -[A^a_b]^{-1} \cdot [A^a_\mu] \cdot [A^\nu_\mu]^{-1} & [A^a_b]^{-1} \end{pmatrix}. \tag{3.5b}$$

Here $A^a_\mu : U \to \mathbb{K}$ and $[A^\nu_\mu]$ and $[A^a_b]$ are non-degenerate matrix-valued functions on U such that $[A^\nu_\mu]$ is constant on the fibres of E, i.e., for $p \in E$, $A^\nu_\mu(p)$ depends only on $\pi(p) \in M$, which is equivalent to any one of the equations $A^\nu_\mu = B^\nu_\mu \circ \pi$ and $\partial A^\nu_\mu/\partial u^a = 0$, with $[B^\nu_\mu]$ being a nondegenerate matrix-valued function on $\pi(U) \subseteq M$. Obviously, (3.2) corresponds to (3.4) with $e_I = \partial/\partial u^I$, $\tilde{e}_I = \partial/\partial\tilde{u}^I$, and $A^J_I = \partial u^J/\partial\tilde{u}^I$.

All frames $\{\tilde{e}_I\}$ on E connected via (3.4)–(3.5) which are (locally) obtainable from holonomic ones $\{e_I\}$, induced by bundle coordinates, via admissible changes, will be referred as *bundle frames*. Only such frames will be employed in the present work.

3.2. Connection theory

From a number of equivalent definitions of a connection on differentiable manifold [16, Sections 2.1 and 2.2], we shall use the following one.

Definition 3.1. A *connection on a bundle* (E, π, M) is an $n = \dim M$- dimensional distribution Δ^h on E such that, for each $p \in E$ and the *vertical distribution* Δ^v defined by

$$\Delta^v : p \mapsto \Delta^v_p := T_{\iota(p)}(\pi^{-1}(\pi(p))) \cong T_p(\pi^{-1}(\pi(p))) \tag{3.6}$$

with $\iota : \pi^{-1}(\pi(p)) \to E$ being the inclusion mapping, is fulfilled

$$\Delta^v_p \oplus \Delta^h_p = T_p(E), \tag{3.7}$$

where $\Delta^h : p \mapsto \Delta^h_p \subseteq T_p(E)$ and $\oplus$ is the direct sum sign. The distribution Δ^h is called *horizontal* and symbolically we write $\Delta^v \oplus \Delta^h = T(E)$.

A *vector* at a point $p \in E$ (resp. a *vector field* on E) is said to be *vertical* or *horizontal* if it (resp. its value at p) belongs to Δ^v_p or Δ^h_p, respectively, for the given (resp. any) point

p. A vector $Y_p \in T_p(E)$ (resp. vector field $Y \in \mathcal{X}(E)$) is called a *horizontal lift of a vector* $X_{\pi(p)} \in T_{\pi(p)}(M)$ (resp. *vector field* $X \in \mathcal{X}(M)$ on $M = \pi(E)$) if $\pi_*(Y_p) = X_{\pi(p)}$ for the given (resp. any) point $p \in E$. Since $\pi_*|_{\Delta_p^h} : \Delta_p^h \to T_{\pi(p)}(M)$ is a vector space isomorphism for all $p \in E$ [14, Section 1.24], any vector in $T_{\pi(p)}(M)$ (resp. vector field in $\mathcal{X}(M)$) has a unique horizontal lift in $T_p(E)$ (resp. $\mathcal{X}(E)$).

As a result of (3.7), any vector $Y_p \in T_p(E)$ (resp. vector field $Y \in \mathcal{X}(E)$) admits a unique representation $Y_p = Y_p^v \oplus Y_p^h$ (resp. $Y = Y^v \oplus Y^h$) with $Y_p^v \in \Delta_p^v$ and $Y_p^h \in \Delta_p^h$ (resp. $Y^v \in \Delta^v$ and $Y^h \in \Delta^h$). If the distribution $p \mapsto \Delta_p^h$ is differentiable of class C^m, $m \in \mathbb{N} \cup \{0, \infty, \omega\}$, it is said that the *connection* Δ^h *is* (*differentiable*) *of class* C^m. A connection Δ^h is of class C^m if and only if, for every C^m vector field Y on E, the vertical Y^v and horizontal Y^h vector fields are of class C^m.

Let us now look on a connections Δ^h on a bundle (E, π, M) from a view point of (local) frames and their dual coframes on E. Let $\{e_\mu\}$ be a basis for Δ^h, i.e. $e_\mu \in \Delta^h$ and $\{e_\mu|_p\}$ is a basis for Δ_p^h for all $p \in E$.

Definition 3.2. A frame $\{e_I\}$ in $T(E)$ over E is called *specialized* for a connection Δ^h if the first $n = \dim M$ of its vector fields $\{e_\mu\}$ form a basis for the horizontal distribution Δ^h and its last $r = \dim \pi^{-1}(x)$, $x \in M$, vector fields $\{e_a\}$ form a basis for the vertical distribution Δ^v.

It is a simple, but important, fact that the specialized frames are (up to renumbering) the most general ones which respect the splitting of $T(E)$ into vertical and horizontal components. Suppose $\{e_I\}$ is a specialized frame. Then the general element of the set of all specialized frames is (see (3.4))

$$(\bar{e}_\mu, \bar{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} A_\mu^\nu & 0 \\ 0 & A_a^b \end{bmatrix} = (A_\mu^\nu e_\nu, A_a^b e_b), \tag{3.8}$$

where $[A_\mu^\nu]_{\mu,\nu=1}^n$ and $[A_a^b]_{a,b=n+1}^{n+r}$ are non-degenerate matrix-valued functions on E, which are constant on the fibres of (E, π, M), i.e. we can set $A_\mu^\nu = B_\mu^\nu \circ \pi$ and $A_a^b = B_a^b \circ \pi$ for some non-degenerate matrix-valued functions $[B_\mu^\nu]$ and $[B_a^b]$ on M.

Since $\pi_*|_{\Delta^h} : \{X \in \Delta^h\} \to \mathcal{X}(M)$ is an isomorphism, any basis $\{\varepsilon_\mu\}$ *for* Δ^h defines a basis $\{E_\mu\}$ of $\mathcal{X}(M)$ such that

$$E_\mu = \pi_*|_{\Delta^h}(\varepsilon_\mu), \tag{3.9}$$

and v.v., a basis $\{E_\mu\}$ for $\mathcal{X}(M)$ induces a basis $\{\varepsilon_\mu\}$ for Δ^h via

$$\varepsilon_\mu = (\pi_*|_{\Delta^h})^{-1}(E_\mu). \tag{3.10}$$

Thus a ‘horizontal’ change

$$\varepsilon_\mu \mapsto \bar{\varepsilon}_\mu = (B_\mu^\nu \circ \pi)\varepsilon_\nu, \tag{3.11}$$

which is independent of a ‘vertical’ one given by

$$\varepsilon_a \mapsto \bar{\varepsilon}_a = (B_a^b \circ \pi)\varepsilon_b \tag{3.12}$$

with $\{\varepsilon_a\}$ being a basis for Δ^v, is equivalent to the transformation

$$E_\mu \mapsto \bar{E}_\mu = B_\mu^\nu E_\nu \tag{3.13}$$

of the basis $\{E_\mu\}$ for $\mathcal{X}(M)$, related via (3.9) to the basis $\{\varepsilon_\mu\}$ for Δ^h. Here $[B_\mu^\nu]$ and $[B_a^b]$ are non-degenerate matrix-valued functions on M.

As $\pi_*(\varepsilon_a) = 0 \in \mathcal{X}(M)$, the 'vertical' transformations (3.12) do not admit interpretation analogous to the 'horizontal' ones (3.11). However, in a case of a *vector* bundle (E, π, M), they are tantamount to changes of frames in the bundle space E, i.e. of the bases for $\mathrm{Sec}\,(E, \pi, M)$. To show this, define a mapping v by

$$\begin{aligned} &v : \mathrm{Sec}\,(E,\pi,M) \to \{\text{vector fields in } \Delta^v\},\\ &v : Y \mapsto Y^v : p \mapsto Y^v|_p := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (p + tY_{\pi(p)}), \end{aligned} \tag{3.14}$$

i.e. v sends a section $Y \in \mathrm{Sec}\,(E, \pi, M)$ of the vector bundle (E, π, M) to a vector field $v(Y) =: Y^v \in \mathcal{X}(E)$ such that Y^v at $p \in E$ is the vector tangent to the path $t \mapsto p + tY_{\pi(p)}$, $t \in \mathbb{R}$, at the point p, that is at $t = 0$; since $\pi(p + tY_{\pi(p)}) \equiv \pi(p)$ for all $t \in \mathbb{R}$, due to $p \in \pi^{-1}(\pi(p))$ and $Y_{\pi(p)} \in \pi^{-1}(\pi(p))$, we have $\pi_*((\mathrm{d}/\mathrm{d}t)|_{t=0}(p + tY_{\pi(p)})) = 0$ which means that $Y_p^v \in \Delta_p^v$ for all $p \in E$, i.e. Y^v is a vertical vector field on E. Since the mapping v is a linear isomorphism [14], the sections

$$E_a = v^{-1}(\varepsilon_a) \tag{3.15}$$

form a basis for $\mathrm{Sec}\,(E, \pi, M)$ as the vertical vector fields ε_a form a basis for Δ^v. Conversely, any basis $\{E_a\}$ for the sections of (E, π, M) induces a basis $\{\varepsilon_a\}$ for Δ^v such that

$$\varepsilon_a = v(E_a). \tag{3.16}$$

As v and v^{-1} are linear, the change (3.12) is equivalent to the transformation

$$E_a \mapsto \bar{E}_a = B_a^b E_b \tag{3.17}$$

of the frame $\{E_a\}$ in E related to $\{\varepsilon_a\}$ via (3.15). In this way, we see that *there is a bijective correspondence between the set of specialized frames* $\{\varepsilon_I\} = \{\varepsilon_\mu, \varepsilon_a\}$ *on a vector bundle* (E, π, M) *and the set of pairs* $(\{E_\mu\}, \{E_a\})$ *of frames* $\{E_\mu\}$ *on* $T(M)$ *over* M *and* $\{E_a\}$ *on* E *over* M.[5] Since conceptually the frames in $T(M)$ and E are easier to be understood and in some cases have a direct physical interpretation, one often works with the pair $(\{E_\mu = \pi_*|_{\Delta^h}(\varepsilon_\mu)\}, \{E_a = v^{-1}(\varepsilon_a)\})$ of frames instead with a specialized frame $\{\varepsilon_I\} = \{\varepsilon_\mu, \varepsilon_a\}$; for instance $\{E_\mu\}$ and $\{E_a\}$ can be completely arbitrary frames in $T(M)$ and E, respectively, while the specialized frames represent only a particular class of frames in $T(E)$.

One can *mutatis mutandis* localize the above considerations when M is replaced with an open subset U_M in M and E is replaced with $U = \pi^{-1}(U_M)$. Such a localization is important

[5] It should be mentioned the evident fact that a frame $\{E_\mu\}$ in $T(M)$ over M is also a basis for the module $\mathcal{X}(M)$ of vector fields over M and hence is a basis for the set $\mathrm{Sec}\,(T(M), \pi_T, M)$ of section of the bundle tangent to M, due to $\mathcal{X}(M) = \mathrm{Sec}\,(T(M), \pi_T, M)$. Similarly, a frame $\{E_a\}$ on E over M is a basis for the set $\mathrm{Sec}\,(E, \pi, M)$ of sections of the vector bundle (E, π, M).

when the bases/frames considered are connected with some local coordinates or when they should be smooth.[6]

Let $\{e_I\}$ be a frame in $T(E)$ defined over an open set $U \subseteq E$ and such that $\{e_a|_p\}$ is a *basis for the space* $T_p(\pi^{-1}(\pi(p)))$ *tangent to the fibre through* $p \in U$. Then we can write the expansion

$$(e^U_\mu, e^U_a) = (D^\nu_\mu e_\mu + D^a_\mu e_a, D^b_a e_b) = (e_\nu, e_b) \cdot \begin{pmatrix} [D^\nu_\mu] & 0 \\ [D^b_\mu] & [D^b_a] \end{pmatrix}, \tag{3.18}$$

where $\{e^U_I\}$ is a *specialized* frame in $T(U)$, $[D^\nu_\mu]$ and $[D^b_a]$ are non-degenerate matrix-valued functions on U, and $D^a_\mu : U \to \mathbb{K}$.

Definition 3.3. The specialized frame $\{X_I\}$ over U in $T(U)$, obtained from (3.18) via an admissible transformation (3.4) with matrix

$$A = \begin{pmatrix} [D^\mu_\nu]^{-1} & 0 \\ 0 & [D^a_b]^{-1} \end{pmatrix},$$

is called *adapted to the frame* $\{e_I\}$ *for* Δ^h.[7]

The frame $\{X_I\}$ adapted to $\{e_I\}$ is independent of the choice of the specialized frame $\{e^U_I\}$ in (3.18) and can alternatively be defined by $X_\mu = (\pi_*|_{\Delta^h})^{-1} \circ \pi_*(e_\mu)$ and $X_a = e_a$.

If $\{u^I\}$ are bundle coordinates on U, the frame $\{X_I\}$ adapted to the coordinate frame $\{\partial/\partial u^I\}$ is said to be *adapted to the coordinates* $\{u^I\}$.

According to (3.4), the adapted frame $\{X_I\} = \{X_\mu, X_a\}$ is given by the equation

$$(X_\mu, X_a) = (e_\nu, e_b) \cdot \begin{bmatrix} \delta^\nu_\mu & 0 \\ +\Gamma^b_\mu & \delta^b_a \end{bmatrix} = (e_\mu + \Gamma^b_\mu e_b, e_a), \tag{3.19}$$

where the functions $\Gamma^a_\mu : U \to \mathbb{K}$, called (*2-index*) *coefficients of* Δ^h in $\{X_I\}$, are defined by

$$[\Gamma^a_\mu] := +[D^a_\nu] \cdot [D^\nu_\mu]^{-1}. \tag{3.20}$$

A change $\{e_I\} \mapsto \{\tilde{e}_I\}$ with

$$(\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \begin{pmatrix} [A^\nu_\mu] & 0 \\ [A^b_\mu] & [A^b_a] \end{pmatrix} = (A^\nu_\mu e_\nu + A^b_\mu e_b, A^b_a e_b), \tag{3.21}$$

where $[A^\nu_\mu]$ and $[A^b_a]$ are non-degenerate matrix-valued functions on U, which are constant on the fibres of (E, π, M), and $A^b_\mu : U \to \mathbb{K}$, entails the transformations (see (3.18)–(3.20))

[6] Recall, not every manifold admits a *global* nowhere vanishing C^m, $m \geq 0$, vector field (see [15] or [17, Section 4.24]); e.g. such are the even-dimensional spheres $\mathbb{S}^{2k}$, $k \in \mathbb{N}$, in Euclidean space.

[7] Recall, here and below the adapted frames are defined only with respect to frames $\{e_I\} = \{e_\mu, e_a\}$ such that $\{e_a\}$ is a basis for the vertical distribution Δ^v over U, i.e. $\{e_a|_p\}$ is a basis for Δ^v_p for all $p \in U$. Since Δ^v is integrable, the relation $e_a \in \Delta^v$ for all $a = n+1, \ldots, n+r$ implies $[e_a, e_b]_- \in \Delta^v$ for all $a, b = n+1, \ldots, n+r$.

$$
\begin{aligned}
(X_\mu, X_a) \mapsto (\tilde{X}_\mu, \tilde{X}_a) &= (\tilde{e}_\mu + \tilde{\Gamma}^b_\mu \tilde{e}_b, \tilde{e}_a) = (A^\nu_\mu X_\nu, A^b_a X_b) \\
&= (X_\nu, X_b) \cdot \begin{bmatrix} A^\nu_\mu & 0 \\ 0 & A^b_a \end{bmatrix},
\end{aligned}
\tag{3.22}
$$

$$
\Gamma^a_\mu \mapsto \tilde{\Gamma}^a_\mu = ([A^c_d]^{-1})^a_b (\Gamma^b_\nu A^\nu_\mu - A^b_\mu)
\tag{3.23}
$$

of the frame $\{X_I\}$ adapted to $\{e_I\}$ and of the coefficients Γ^a_μ of Δ^h in $\{X_I\}$, i.e. $\{\tilde{X}_I\}$ is the frame adapted to $\{\tilde{e}_I\}$ and $\tilde{\Gamma}^a_\mu$ are the coefficients of Δ^h in $\{\tilde{X}_I\}$.

Note 3.1. If $\{e_I\}$ and $\{\tilde{e}_I\}$ are adapted, then $A^b_\mu = 0$. If $\{Y_I\}$ is a specialized frame, it is adapted to any frame $\{e_\mu = A^\nu_\mu Y_\nu, e_a = A^b_a Y_b\}$ and hence any specialized frame can be considered as an adapted one; in particular, any specialized frame is a frame adapted to itself. Obviously, the coefficients of a connection identically vanish in a given specialized frame considered as an adapted one. This leads to the concept of a *normal frame* to which is devoted the present paper. Besides, from the above observation follows that the set of adapted frames coincides with the one of specialized frames.

In particular, if $\{u^I\}$ and $\{\tilde{u}^I\}$ are local bundle coordinates with non-empty intersection of their domains, we can set

$$
e_I = \frac{\partial}{\partial u^I}, \qquad \tilde{e}_I = \frac{\partial}{\partial \tilde{u}^I},
\tag{3.24}
$$

which entails

$$
A^\nu_\mu = \frac{\partial u^\nu}{\partial \tilde{u}^\mu}, \qquad A^b_\mu = \frac{\partial u^b}{\partial \tilde{u}^\mu}, \qquad A^b_a = \frac{\partial u^b}{\partial \tilde{u}^a}.
\tag{3.25}
$$

So, when the holonomic choice (3.24) is made, the transformation (3.23) reduces to

$$
\Gamma^a_\mu \mapsto \tilde{\Gamma}^a_\mu = \left(\frac{\partial \tilde{u}^a}{\partial u^b} \Gamma^b_\nu + \frac{\partial \tilde{u}^a}{\partial u^\nu} \right) \frac{\partial u^\nu}{\partial \tilde{u}^\mu}.
\tag{3.26}
$$

Let (E, π, M) be a *vector* bundle. According to the above-said in this section, any *adapted* frame $\{X_I\} = \{X_\mu, X_a\}$ in $T(E)$ is equivalent to a pair of frames in $T(M)$ and E according to

$$
\{X_\mu, X_a\} \leftrightarrow (\{E_\mu = \pi_*|_{\Delta^h}(X_\mu)\}, \{E_a = v^{-1}(X_a)\}).
\tag{3.27}
$$

Suppose $\{X_I\}$ and $\{\tilde{X}_I\}$ are two adapted frames. Then they are connected by (cf. (3.8) and (3.22))

$$
\tilde{X}_\mu = (B^\nu_\mu \circ \pi) X_\nu, \qquad \tilde{X}_a = (B^b_a \circ \pi) X_b,
\tag{3.28}
$$

where $[B^\nu_\mu]$ and $[B^b_a]$ are some non-degenerate matrix-valued functions on M. The pairs of frames corresponding to them, in accordance with (3.27), are related via

$$
\tilde{E}_\mu = B^\nu_\mu E_\nu, \qquad \tilde{E}_a = B^b_a E_b
\tag{3.29}
$$

and vice versa.

In the vector bundles are used, as we shall do below, the so-called vector bundle coordinates which are linear on their fibres and are constructed as follows (cf. [18, p. 30]).

Let $\{e_a\}$ be a frame in E over a subset $U_M \subseteq M$, i.e. $\{e_a(x)\}$ to be a basis in $\pi^{-1}(x)$ for all $x \in U_M$. Then, for each $p \in \pi^{-1}(U_M)$, we have a unique expansion $p = p^a e_a(\pi(p))$ for some numbers $p^a \in \mathbb{K}$. The *vector fibre coordinates* $\{u^a\}$ on $\pi^{-1}(U_M)$ induced (generated) by the frame $\{e_a\}$ are defined via $u^a(p) := p^a$ and hence can be identified with the elements of the coframe $\{e^a\}$ dual to $\{e_a\}$, i.e. $u^a = e^a$. Conversely, if $\{u^I\}$ are coordinates on $\pi^{-1}(U_M)$ for some $U_M \subseteq M$ which are linear on the fibres over U_M, then there is a unique frame $\{e_a\}$ in $\pi^{-1}(U_M)$ which generates $\{u^a\}$ as just described; indeed, considering $u^{n+1}, \dots, u^{n+r}$ as 1-forms on $\pi^{-1}(U_M)$, one should define the frame $\{e_a\}$ required as a one whose dual is $\{u^a\}$, i.e. via the conditions $u^a(e_b) = \delta^a_b$.

A collection $\{u^I\}$ of basic coordinates $\{u^\mu\}$ and vector fibre coordinates $\{u^a\}$ on $\pi^{-1}(U_M)$ is called *vector bundle coordinates* on $\pi^{-1}(U_M)$. Only such coordinates on E will be employed in what follows.

The following result gives a full local description of the linear connections on vector bundles.[8] The importance of these connection comes from the fact that they are compatible with the linear structure of the vector bundles and are the most widely used connections.

Proposition 3.1. *Let Δ^h be a linear connection on a vector bundle (E, π, M) and $\{X_I\}$ be the frame adapted for Δ^h to a frame $\{e_I\}$ such that $\{e_a\}$ is a basis for Δ^v. Let $\{u^I\} = \{u^\mu, u^a\}$ be vector bundle coordinates on $U \subseteq E$. Suppose that the frame $\{e_I\}$, to which $\{X_I\}$ is adapted to, is such that*

$$\begin{aligned}(e_\mu, e_a)|_U &= (\partial_\nu, \partial_b) \cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ (B^b_{c\mu} \circ \pi) \cdot u^c & B^b_a \circ \pi \end{bmatrix} \\ &= ((B^\nu_\mu \circ \pi)\partial_\nu + ((B^b_{c\mu} \circ \pi) \cdot u^c)\partial_b, (B^b_a \circ \pi)\partial_b),\end{aligned} \tag{3.33}$$

[8] A connection on a vector bundle is linear if the parallel transport generated by it is a linear mapping [19]. More precisely, we recall the following two definitions.

Definition 3.4. Let $\gamma : [\sigma, \tau] \to M$, with $\sigma, \tau \in \mathbb{R}$ and $\sigma \le \tau$, and $\bar{\gamma}_p$ be the unique horizontal lift of γ in E passing through $p \in \pi^{-1}(\gamma([\sigma, \tau]))$. The *parallel transport* (*translation*, *displacement*) generated by (assigned to, defined by) a connection Δ^h is a mapping $\mathsf{P} : \gamma \mapsto \mathsf{P}^\gamma$, assigning to the path γ a mapping

$$\mathsf{P}^\gamma : \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau)), \qquad \gamma : [\sigma, \tau] \to M \tag{3.30}$$

such that, for each $p \in \pi^{-1}(\gamma(\sigma))$,

$$\mathsf{P}^\gamma(p) := \bar{\gamma}_p(\tau). \tag{3.31}$$

Definition 3.5. A connection on a vector bundle is called *linear* if the assigned to it parallel transport is a linear mapping along every path in the base space, i.e. if the mapping (3.30) is linear for all paths $\gamma : [\sigma, \tau] \to M$ in the base, viz.

$$\mathsf{P}^\gamma(\rho X) = \rho \mathsf{P}^\gamma(X), \tag{3.32a}$$

$$\mathsf{P}^\gamma(X + Y) = \mathsf{P}^\gamma(X) + \mathsf{P}^\gamma(Y), \tag{3.32b}$$

where $\rho \in \mathbb{K}$ and $X, Y \in \pi^{-1}(\gamma(\sigma))$.

where $\partial_I := \partial/\partial u^I$, $[B^\nu_\mu]$ *and* $[B^b_a]$ *are non-degenerate matrix-valued functions on* $\pi(U)$, *and* $B^b_{c\mu} : \pi(U) \to \mathbb{K}$. *Then the 2-index coefficients* Γ^a_μ *of* Δ^h *in* $\{X_I\}$ *have the representation*

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu} \circ \pi) \cdot u^b \tag{3.34}$$

on U *for some functions* $\Gamma^a_{b\mu} : \pi(U) \to \mathbb{K}$, *called 3-index coefficients of* Δ^h *in* $\{X_I\}$.

Remark 3.1. The representation (3.34) is not valid for frames more general than the ones given by (3.33). Precisely, Eq. (3.34) is valid if and only if (3.33) holds for some local coordinates $\{u^I\}$ on U—see (3.23).

Remark 3.2. Since the vector fibre coordinates u^a are 1-forms on U, the 2-index coefficients (3.34) of a linear connection are also 1-forms on the bundle space.

Lemma 3.1 (cf. [19, p. 27]). *Let* (E, π, M) *be a vector bundle,* $\{u^I\}$ *be vector bundle coordinates on an open set* $U \subseteq E$, *and* Δ^h *be a connection on it described in the frame* $\{X_I\}$, *adapted to* $\{u^I\}$, *by its 2-index coefficients* Γ^a_μ. *The connection* Δ^h *is linear if and only if, for each* $p \in U$,

$$\Gamma^a_\mu(p) = -\Gamma^a_{b\mu}(\pi(p))u^b(p) = -((\Gamma^a_{b\mu} \circ \pi) \cdot u^b)(p), \tag{3.35}$$

where $\Gamma^a_{b\mu} : \pi(U) \to \mathbb{K}$ *are some functions on the set* $\pi(U) \subseteq M$ *and the minus sign before* $\Gamma^a_{b\mu}$ *in* (3.35) *is conventional.*

Proof. Take a C^1 path $\gamma : [\sigma, \tau] \to \pi(U)$ and consider the parallel transport equation

$$\frac{\mathrm{d}\bar{\gamma}^a_p(t)}{\mathrm{d}t} = \Gamma^a_\mu(\bar{\gamma}_p(t))\dot{\gamma}^\mu(t), \tag{3.36}$$

where $\bar{\gamma}_p : [\sigma, \tau] \to U$ is the horizontal lift of γ through $p \in \pi^{-1}(\gamma(\sigma))$, $\bar{\gamma}^a := u^a \circ \bar{\gamma}$, and $\dot{\gamma}^\mu(t) = \mathrm{d}(x^\mu \circ \gamma(t))/\mathrm{d}t = \mathrm{d}(u^\mu \circ \bar{\gamma}(t))/\mathrm{d}t$ as $u^\mu = x^\mu \circ \pi$ for some coordinates $\{x^\mu\}$ on $\pi(U)$.[9]

Sufficiency. If (3.35) holds, (3.36) is transformed into

$$\frac{\mathrm{d}\bar{\gamma}^a_p(t)}{\mathrm{d}t} = -\Gamma^a_{b\mu}(\gamma(t))\bar{\gamma}^b_p(t)\dot{\gamma}^\mu(t), \tag{3.39}$$

[9] If $\{\omega^I\} = \{\omega^\mu = \mathrm{d}u^\mu, \omega^a = \mathrm{d}u^a - \Gamma^a_\mu \mathrm{d}u^\mu\}$ is the frame dual to $\{X_I\}$, then the path $\dot{\gamma}_p$ is given as the unique solution of the initial value problem

$$\omega^a(\dot{\bar{\gamma}}_p) = 0, \tag{3.37a}$$

$$\bar{\gamma}_p(t_0) = p, \tag{3.37b}$$

which is tantamount to

$$\frac{\mathrm{d}(u^a \circ \bar{\gamma}_p(t))}{\mathrm{d}t} - \Gamma^a_\mu(\bar{\gamma}_p(t))\frac{\mathrm{d}(x^\mu \circ \gamma(t))}{\mathrm{d}t} = 0, \tag{3.38a}$$

$$u^I(\bar{\gamma}_p(t_0)) = u^I(p). \tag{3.38b}$$

Eq. (3.38a) is (a form of) the parallel transport equation along γ.

which is a system of r linear first order ordinary differential equations for the r functions $\bar{\gamma}_p^{n+1}, \dots, \bar{\gamma}_p^{n+r}$. According to the general theorems of existence and uniqueness of the solutions of such systems [20], it has a unique solution

$$\bar{\gamma}_p^a(t) = Y_b^a(t) p^b \tag{3.40}$$

satisfying the initial condition $\bar{\gamma}_p^a(\sigma) = u^a(p) =: p^a$, where $Y = [Y_b^a]$ is the fundamental solution of (3.39), i.e.

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = -[\Gamma_{b\mu}^a(\gamma(t))\dot{\gamma}^\mu(t)]_{a,b=n+1}^{n+r} \cdot Y(t) \quad Y(\sigma) = \mathbb{1}_{r\times r} = [\delta_b^a]. \tag{3.41}$$

The linearity of (3.30) in p follows from (3.40) for $t = \tau$.

Necessity. Suppose (3.30) is linear in p for all paths $\gamma : [\sigma, \tau] \to \pi(U)$. Then $\bar{\gamma}_p(t) := \mathsf{P}^{\gamma|[\sigma,t]}(p)$ is the horizontal lift of $\gamma|[\sigma, t]$ through p and (cf. (3.40)) $\bar{\gamma}_p^a(t) = A_b^a(\gamma(t))p^b$ for some C^1 functions $A_b^a : \pi(U) \to \mathbb{K}$. The substitution of this equation in (3.36) results into

$$\left.\frac{\partial A_b^a(x)}{\partial x^\mu}\right|_{x=\gamma(t)=\pi(\bar{\gamma}_p(t))} \cdot \dot{\gamma}^\mu p^b = \Gamma_\mu^a(\bar{\gamma}_p(t))\dot{\gamma}^\mu(t).$$

Since $\gamma : [\sigma, \tau] \to M$, we get Eq. (3.35) from here, for $t = \sigma$, with $\Gamma_{b\mu}^a(x) = -(\partial A_b^a(x)/\partial x^\mu)$ for $x \in \pi(U)$. □

Proof of proposition 3.1. If $e_I = \partial/\partial u^I$ for some bundle coordinates $\{u^I\}$ on E, the proposition coincides with Lemma 3.1. Writing (3.23) for the transformation $\{\partial_I\} \mapsto \{e_I\}$, with $\{e_I\}$ given by (3.33), we get (3.34) with

$$\Gamma_{b\mu}^a = ([B_d^e]^{-1})_c^a({}^\partial\Gamma_{b\nu}^c B_\mu^\nu + B_{b\mu}^c),$$

where ${}^\partial\Gamma_{b\nu}^a$ are the 3-index coefficients of Δ^h in the frame adapted to the coordinates $\{u^I\}$. □

Let $\{X_I\}$ and $\{\tilde{X}_I\}$ be frames adapted to $\{e_I\}$ and $\{\tilde{e}_I\}$, respectively, such that (cf. (3.33))

$$(\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} B_\mu^\nu \circ \pi & 0 \\ (B_{c\mu}^b \circ \pi) \cdot u^c & B_a^b \circ \pi \end{bmatrix}, \tag{3.42}$$

and Δ^h admits 3-index coefficients in $\{X_I\}$ and $\{\tilde{X}_I\}$, which means that $\{e_I\}$ and $\{\tilde{e}_I\}$ are obtainable from the frames $\{\partial/\partial u^I\}$ and $\{\partial/\partial \tilde{u}^I\}$, respectively, for some bundle coordinates $\{u^I\}$ and $\{\tilde{u}^I\}$ via equations like (3.33) (with $\tilde{e}_I$ for e_I and $\tilde{\partial}_I$ for ∂_I in the letter case) in which the B's need not be the same as in (3.42).[10] Then, due to (3.23) and (3.34), the 3-index coefficients $\Gamma_{b\mu}^a$ and $\tilde{\Gamma}_{b\mu}^a$ of Δ^h in respectively $\{X_I\}$ and $\{\tilde{X}_I\}$ are connected by (see also Footnote 10)

$$\tilde{\Gamma}_{b\mu}^a = ([B_f^e]^{-1})_c^a(\Gamma_{d\nu}^c B_\mu^\nu + B_{d\mu}^c)B_b^d. \tag{3.43}$$

[10] Notice, from (3.42) follows that the vector fibre coordinates $\{u^a\}$ and $\{\tilde{u}^a\}$ are connected by $u^a = (B_b^a \circ \pi) \cdot \tilde{u}^b$.

It can easily be checked that the transformation $\{e_I\} \mapsto \{\tilde{e}_I\}$, with $\{\tilde{e}_I\}$ given by (3.42), is the most general one that preserves the existence of 3-index coefficients of Δ^h provided they exist in $\{e_I\}$ in a sense that, if $\{e_I\}$ is given by (3.33) (which leads to (3.34)) and $\{\tilde{e}^I\}$ is given by (3.42), then there exist vector bundle coordinates $\{\tilde{u}^I\}$ which generate $\{\tilde{e}^I\}$ according to (3.33) with $\tilde{e}_I$ for e_I, $\tilde{\partial}_I$ for ∂_I and some B's, which leads to (3.34) with $\tilde{\Gamma}$ for Γ and $\tilde{u}$ for u. Introducing the matrices $\Gamma_\mu := [\Gamma^a_{b\mu}]^{n+r}_{a,b=n+1}$, $\tilde{\Gamma}_\mu := [\tilde{\Gamma}^a_{b\mu}]^{n+r}_{a,b=n+1}$, $B := [B^a_b]$, and $B_\mu := [B^a_{b\mu}]$, we rewrite (3.43) as

$$\tilde{\Gamma}_\mu = B^{-1} \cdot (\Gamma_\nu B^\nu_\mu + B_\mu) \cdot B. \tag{3.43$'$}$$

A little below (see the text after Eq. (3.45)), we shall prove that the compatibility of the developed formalism with the theory of covariant derivatives requires further restrictions on the general transformed frames (3.21) to the ones given by (3.42) with

$$B_\mu = \tilde{E}_\mu(B) \cdot B^{-1} = B^\nu_\mu E_\nu(B) \cdot B^{-1}, \tag{3.44}$$

where $\tilde{E}_\mu := \pi_*|_{\Delta^h}(\tilde{X}_\mu) = \pi_*|_{\Delta^h}((B^\nu_\mu \circ \pi)X_\nu) = B^\nu_\mu E_\nu$. In this case, (3.43$'$) reduces to

$$\tilde{\Gamma}_\mu = B^\nu_\mu B^{-1} \cdot (\Gamma_\nu \cdot B + E_\nu(B)) = B^\nu_\mu (B^{-1} \cdot \Gamma_\nu - E_\nu(B^{-1})) \cdot B. \tag{3.45}$$

At last, a few words on the covariant derivatives operators ∇ are in order. Without lost of generality, we define such an operator

$$\nabla : \mathcal{X}(M) \times \mathrm{Sec}^1(E, \pi, M) \to \mathrm{Sec}^0(E, \pi, M), \qquad \nabla : (F, Y) \mapsto \nabla_F Y \tag{3.46}$$

via the equations

$$\nabla_{F+G} Y = \nabla_F Y + \nabla_G Y, \tag{3.47a}$$

$$\nabla_{fF} Y = f \nabla_F Y, \tag{3.47b}$$

$$\nabla_F (Y + Z) = \nabla_F Y + \nabla_F Z, \tag{3.47c}$$

$$\nabla_F (fY) = F(f) \cdot Y + f \cdot \nabla_F Y, \tag{3.47d}$$

where $F, G \in \mathcal{X}(M)$, $Y, Z \in \mathrm{Sec}^1(E, \pi, M)$, and $f : M \to \mathbb{K}$ is a C^1 function. Suppose $\{E_\mu\}$ is a basis for $\mathcal{X}(M)$ and $\{E_a\}$ is a one for $\mathrm{Sec}^1(E, \pi, M)$. Define the *components* $\Gamma^a_{b\mu} : M \to \mathbb{K}$ of ∇ in the pair of frames $(\{E_\mu\}, \{E_a\})$ by

$$\nabla_{E_\mu}(E_b) = \Gamma^a_{b\mu} E_a. \tag{3.48}$$

Then (3.47) imply

$$\nabla_F Y = F^\mu (E_\mu(Y^a) + \Gamma^a_{b\mu} Y^b) E_a$$

for $F = F^\mu E_\mu \in \mathcal{X}(M)$ and $Y = Y^a E_a \in \mathrm{Sec}^1(E, \pi, M)$. A change $(\{E_\mu\}, \{E_a\}) \mapsto (\{\tilde{E}_\mu\}, \{\tilde{E}_a\})$, given via (3.29), entails

$$\Gamma^a_{b\mu} \mapsto \tilde{\Gamma}^a_{b\mu} = B^\nu_\mu ([B^e_f]^{-1})^a_c (\Gamma^c_{d\nu} B^d_b + E_\nu(B^c_b)), \tag{3.49}$$

as a result of (3.48). In a more compact matrix form, the last result reads

$$\tilde{\Gamma}_\mu = B^\nu_\mu B^{-1} \cdot (\Gamma_\nu \cdot B + E_\nu(B)) \tag{3.49$'$}$$

with $\Gamma_\mu := [\Gamma^a_{b\mu}]$, $\tilde{\Gamma}_\mu := [\tilde{\Gamma}^a_{b\mu}]$, and $B := [B^a_b]$.

Thus, if we identify the 3-index coefficients of Δ^h, defined by (3.34), with the components of ∇, defined by (3.48),[11] then the quantities (3.43′) and (3.49′) must coincide, which immediately leads to the equality (3.44). Therefore

$$(e_\mu, e_a) \mapsto (\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \left. \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ ((B^\nu_\mu E_\nu (B^b_d)(B^{-1})^d_c) \circ \pi) u^c & B^b_a \circ \pi \end{bmatrix} \right|_{B=[B^b_a]} \tag{3.50}$$

is the most general transformation between frames in $T(E)$ such that the frames adapted to them are compatible with the linear connection and the covariant derivative corresponding to it. In particular, such are all frames $\{\partial/\partial u^I\}$ in $T(E)$ induced by some vector bundle coordinates $\{u^I\}$ on E as the vector fibre coordinates transform in a linear way like $u^a \mapsto \tilde{u}^a = (B^a_b \circ \pi) \cdot u^b$; the rest members of the class of frames mentioned are obtained from them via (3.50) with $e_I = \partial/\partial u^I$ and some non-degenerate matrix-valued functions $[B^\nu_\mu]$ and B.

If $\{X_I\}$ (resp. $\{\tilde{X}_I\}$) is the frame adapted to a frame $\{e_I\}$ (resp. $\{\tilde{e}_I\}$), then the change $\{e_I\} \mapsto \{\tilde{e}_I\}$, given by (3.50), entails $\{X_I\} \mapsto \{\tilde{X}_I\}$ with $\{\tilde{X}_I\}$ given by (3.28) (see (3.21) and (3.22)). Since the last transformation is tantamount to the change

$$(\{E_\mu\}, \{E_a\}) \mapsto (\{\tilde{E}_\mu\}, \{\tilde{E}_a\}) \tag{3.51}$$

of the basis of $\mathcal{X}(M) \times \mathrm{Sec}\,(E, \pi, M)$ corresponding to $\{X_I\}$ via (3.27)–(3.29)), we can say that the transition (3.51) induces the change (3.49) of the 3-index coefficients of the connection Δ^h. Exactly the same is the situation one meets in the literature [21,13,14] when covariant derivatives are considered (and identified with connections).

Regardless that the change (3.50) of the frames in $T(E)$ looks quite special, it is the most general one that, through (3.22) and (3.27), is equivalent to an arbitrary change (3.51) of a basis in $\mathcal{X}(M) \times \mathrm{Sec}\,(E, \pi, M)$, i.e. of a pair of frames in $T(M)$ and E.

The above results, concerning linear connections on vector bundles, can be generalized for affine connections on vector bundles.[12] For instance, the analogue of Propositions 3.1 reads.

[11] Such an identification is justified by the definition of ∇ via the parallel transport assigned to Δ^h or via a projection, generated by Δ^h, of a suitable Lie derivative on $\mathfrak{X}(E)$—see [19].

[12] Usually the affine connections are defined on affine bundles [21,16]. In vector bundles they can be introduced as follows.

Definition 3.6. A connection on a vector bundle is termed *affine connection* if the assigned to it parallel transport $\mathsf{P} : \gamma \mapsto \mathsf{P}^\gamma : \pi^{-1}(\gamma(\sigma)) \to \pi^{-1}(\gamma(\tau))$ is an affine mapping along all paths $\gamma : [\sigma, \tau] \to M$ in the base space, i.e.

$$\mathsf{P}^\gamma(\rho X) = \rho \mathsf{P}^\gamma(X) + (1 - \rho)\mathsf{P}^\gamma(\mathbf{0}), \tag{3.52a}$$

$$\mathsf{P}^\gamma(X + Y) = \mathsf{P}^\gamma(X) + \mathsf{P}^\gamma(Y) - \mathsf{P}^\gamma(\mathbf{0}), \tag{3.52b}$$

where $\rho \in \mathbb{K}$, $X, Y \in \pi^{-1}(\gamma(\sigma))$, and $\mathbf{0}$ is the zero vector in the fibre $\pi^{-1}(\gamma(\sigma))$, which is a $\mathbb{K}$-vector space.

An affine connection for which $\mathsf{P}^\gamma(\mathbf{0})$ is the zero vector in $\pi^{-1}(\gamma(\tau))$ is a linear connection and vice versa—see Definition 3.5.

Proposition 3.2. *Let Δ^h be an affine connection on a vector bundle (E,π,M) and $\{X_\mu\}$ be the frame adapted for Δ^h to a frame $\{e_I\}$ such that $\{e_a\}$ is a basis for Δ^v and*

$$\begin{aligned}(e_\mu,e_a)|_U &= (\partial_\nu,\partial_b)\cdot\begin{bmatrix} B^\nu_\mu\circ\pi & 0\\ (B^b_{c\mu}\circ\pi)\cdot u^c & B^b_a\circ\pi\end{bmatrix}\\ &= ((B^\nu_\mu\circ\pi)\partial_\nu + ((B^b_{c\mu}\circ\pi)\cdot u^c)\partial_b, (B^b_a\circ\pi)\partial_b),\end{aligned} \tag{3.53}$$

where $\partial_I := \partial/\partial u^I$ for some local bundle coordinates $\{u^I\}=\{u^\mu=x^\mu\circ\pi, u^b=E^b\}$ on $U\subseteq E$, $[B^\nu_\mu]$ and $[B^b_a]$ are non-degenerate matrix-valued functions on U, and $B^b_{c\mu}:U\to\mathbb{K}$. Then the 2-index coefficients Γ^a_μ of Δ^h in $\{X_I\}$ have the representation

$$\Gamma^a_\mu = -(\Gamma^a_{b\mu}\circ\pi)\cdot u^b + G^a_\mu\circ\pi \tag{3.54}$$

on U for some functions $\Gamma^a_{b\mu}, G^a_\mu : U\to\mathbb{K}$.

Remark 3.3. The representation (3.54) is not valid for frames more general than the ones given by (3.53). Precisely, Eq. (3.54) is valid if and only if (3.53) holds for some local coordinates $\{u^I\}$ on U—see (3.23).

Proof. Writing (3.23) for the transformation $\{\partial_I\}\mapsto\{e_I\}$, with $\{e_I\}$ given by (3.53), we get (3.54) with

$$\Gamma^a_{b\mu} = ([B^e_d]^{-1})^a_c({}^\partial\Gamma^c_{b\nu}B^\nu_\mu + B^c_{b\mu}),\qquad G^a_\mu = ([B^e_d]^{-1})^a_b{}^\partial G^b_\nu B^\nu_\mu,$$

where ${}^\partial\Gamma^a_{b\nu}$ and ${}^\partial G^b_\nu$ are defined via the 2-index coefficients ${}^\partial\Gamma^a_\mu$ of Δ^h in the frame adapted to the coordinates $\{u^I\}$ via ${}^\partial\Gamma^a_\mu = -({}^\partial\Gamma^a_{b\mu}\circ\pi)\cdot E^b + {}^\partial G^a_\mu\circ\pi$, which is the general form of the 2-index coefficients, in such a frame, of an affine connection. The last assertion can be proved similarly to Lemma 3.1. $\square$

Let $\{X_I\}$ and $\{\tilde X_I\}$ be frames adapted to $\{e_I\}$ and $\{\tilde e_I\}$, respectively, with (cf. (3.53))

$$(\tilde e_\mu,\tilde e_a) = (e_\nu,e_b)\cdot\begin{bmatrix} B^\nu_\mu\circ\pi & 0\\ (B^b_{c\mu}\circ\pi)\cdot u^c & B^b_a\circ\pi\end{bmatrix}, \tag{3.55}$$

in which (3.54) holds for Δ^h. Then, due to (3.23) and (3.54), the pairs $(\Gamma^a_{b\mu}, G^a_\mu)$ and $(\tilde\Gamma^a_{b\mu},\tilde G^a_\mu)$ for Δ^h in respectively $\{X_I\}$ and $\{\tilde X_I\}$ are connected by

$$\tilde\Gamma^a_{b\mu} = ([B^e_f]^{-1})^a_c(\Gamma^c_{d\nu}B^\nu_\mu + B^c_{d\mu})B^d_b, \tag{3.56a}$$

$$\tilde G^a_\mu = ([B^e_f]^{-1})^a_b G^b_\nu B^\nu_\mu. \tag{3.56b}$$

From here and (3.43), we conclude that $\Gamma^a_{b\nu}$ are coefficients of a linear connection on the same bundle. We call it *corresponding* to the affine connection under consideration.

Remark 3.4. It can be proved that the transformation $\{e_I\}\mapsto\{\tilde e_I\}$, with $\{\tilde e_I\}$ given by (3.55), is the most general one that preserves the existence of the relation (3.54) for Δ^h provided it holds in $\{e_I\}$.

4. Normal frames: general case

In the theory of linear connections on a manifold, the normal frames are defined as frames in the tangent bundle space in which the connections' (3-index) coefficients vanish on some subset of the manifold [21,14,6–9]. The definition of normal frames for a connection on a vector bundle is practically the same, the only difference being that these frames are in the bundle space, not in the tangent bundle space over the base space [10]. The present section is devoted to the introduction of normal frames for general connections on fibre bundles and some their properties.

To save some space and for brevity, in what follows we shall not indicate explicitly that the frames $\{e_I\} = \{e_\mu, e_a\}$, with respect to which the adapted frames are defined, are such that $\{e_a\}$ is a (local) basis for the vertical distribution Δ^v on the bundle considered.

Definition 4.1. Given a connection Δ^h on a bundle (E, π, M) and a subset $U \subseteq E$. A *frame* $\{X_I\}$ *in* $T(E)$ *adapted to a frame* $\{e_I\}$ *in* $T(E)$ and defined over an open subset V of E containing or equal to U, $V \supseteq U$, is called *normal for* Δ^h *over/on* U (*relative to* $\{e_I\}$) if all (2-index) coefficients Γ^a_μ of Δ^h vanish in it everywhere on U. Respectively, $\{X_I\}$ is *normal for* Δ^h *along a mapping* $g : Q \to E$, $Q \neq \emptyset$, if $\{X_I\}$ is normal for Δ^h over the set $g(Q)$.

Let $\{X_I\}$ be the frame in $T(E)$ adapted to a frame $\{e_I\}$ in $T(E)$ over an open subset $V \subseteq E$. Then the frame $\{\tilde{X}_I\}$ in $T(E)$ adapted to a frame $\{\tilde{e}_I\}$, given by (3.21), in $T(E)$ over the same subset V is normal for Δ^h over $U \subseteq V$ if and only if

$$(A^\nu_\mu \Gamma^b_\nu - A^b_\mu)|_U = 0, \tag{4.1}$$

due to (3.22) and (3.23). Since Γ^b_μ depend only on Δ^h and $\{e_I\}$, the existence of solutions of (4.1), relative to A^ν_μ and A^b_μ, and their properties are completely responsible for the existence and the properties of frames normal for Δ^h over U. For that reason, we call (4.1) the (*system of*) *equation*(*s*) *of the normal frames for* Δ^h *over* U or simply the *normal frame* (*system of*) *equation*(*s*) (for Δ^h over U).

In the most general case, when no additional restrictions on the frames considered are imposed, the normal frames Eq. (4.1) is a system of *nr linear algebraic equations* for $nr + n^2$ variables and, consequently, it has a solution depending on n^2 independent parameters. In particular, if we choose the functions $A^\nu_\mu : U \to \mathbb{K}$ (with $\det[A^\nu_\mu] \neq 0, \infty$) as such parameters, we can write the general solution of (4.1) as

$$(\{A^\nu_\mu\}, \{A^b_\mu\})|_U = (\{A^\nu_\mu\}, \{\Gamma^b_\nu A^\nu_\mu\})|_U. \tag{4.2}$$

It should be noted, Eq. (4.1) or its general solution (4.2) defines the frame $\{\tilde{e}_I\}$ and the frame $\{\tilde{X}_I\}$ adapted to $\{\tilde{e}_I\}$ only on U and leaves them completely arbitrary on $V \setminus U$, if it is not empty.

Proposition 4.1. *Let* $\{X_I\}$ *be the frame adapted to a frame* $\{e_I\}$ *in* $T(V) \subseteq T(E)$ *defined over an open set* $V \subseteq E$ *and* Γ^a_μ *be the coefficients of a connection* Δ^h *in* $\{X_I\}$. *Then all frames* $\{\tilde{X}_I\}$, *normal on* $U \subseteq V$ *for the connection* Δ^h, *are adapted to frames* $\{\tilde{e}_I\}$ *given on*

U by

$$\tilde{e}_\mu|_U = (A^\nu_\mu(e_\nu + \Gamma^b_\nu e_b))|_U \quad \tilde{e}_a|_U = (A^b_a e_b)|_U. \tag{4.3}$$

where $[A^\nu_\mu]$ and $[A^b_a]$ are non-degenerate matrix-valued functions on V which are constant on the fibres of (E, π, M). Moreover, the frame $\{\tilde{X}_I\}$ adapted on V to $\{\tilde{e}_I\}$, given by (4.3) (and hence normal on U), is such that

$$\tilde{X}_\mu|_U = (A^\nu_\mu X_\nu)|_U = \tilde{e}_\mu|_U, \qquad \tilde{X}_a|_U = (A^b_a X_b)|_U = \tilde{e}_a|_U. \tag{4.4}$$

Proof. Apply (3.22), (3.21), and (3.19) for the choice (4.2). □

Eqs. (4.4) are not accidental as it is stated by the following assertion.

Proposition 4.2. *The frame $\{\tilde{X}_I\}$ in $T(E)$ adapted to a frame $\{\tilde{e}_I\}$ in $T(E)$ and defined over an open set $V \subseteq E$ is normal on $U \subseteq V$ if and only if on U is fulfilled*

$$\tilde{X}_I|_U = \tilde{e}_I|_U. \tag{4.5}$$

Proof. Apply (3.19) or (3.22) and Definition 4.1. □

Thus one can equivalently *define the normal frames as adapted frames that coincide on some set with the frames they are adapted to* or as frames (in the tangent bundles space over the bundle space) that coincide on some set with the frames adapted to them.

Since any specialized frame is adapted to itself (see Definition 3.3 and (3.18), with $D^J_I = \delta^J_I$), *the sets of normal, specialized, and adapted frames are identical.*

As we see from Proposition 4.1, which gives a complete description of the normal frames, the theory of normal frames in the most general setting is trivial. It becomes more interesting and richer if the class of frames $\{e_I\}$, with respect to which are defined the adapted frames, is restricted in one or other way. To the theory of normal frames, adapted to such restricted classes of frames in $T(E)$, are devoted the next two sections.

5. Normal frames adapted to holonomic frames

The class of holonomic frames induced by local coordinates on E (see Section 3.2) is the most natural class of frames in $T(E)$ relative to which the adapted, in particular normal, frames are defined. To specify the consideration of the previous section to normal frames adapted to local coordinates on E, we set $e_I = \partial/\partial u^I$ and $\tilde{e}_I = \partial/\partial \tilde{u}^I$, where $\{u^I\}$ and $\{\tilde{u}^I\}$ are local coordinates on E whose domains have a non-empty intersection V and $U \subseteq V$. Then the matrix $[A^J_I]$ in (4.1) is given by (3.3) (as $\{e_I\} \mapsto \{\tilde{e}_I\}$ reduces to (3.2)), so that the normal frame Eq. (4.1) reduces to the *normal coordinates equation* (see also (3.26))

$$\left.\left(\frac{\partial \tilde{u}^a}{\partial u^b}\Gamma^b_\mu + \frac{\partial \tilde{u}^a}{\partial u^\mu}\right)\right|_U = 0, \tag{5.1}$$

due to (3.1), which is a first order system of *nr linear partial differential equations* on U relative to the r unknown functions $\{\tilde{u}^{n+1}, \dots, \tilde{u}^{n+r}\}$.

Since the connection Δ^h is supposed given and fixed, such are its coefficients Γ^b_μ in $\{\partial/\partial u^I\}$. Therefore the existence, uniqueness and other properties of the solutions of (5.1) strongly depend on the set U (which is in the intersection of the domains of the local coordinates $\{u^I\}$ and $\{\tilde{u}^I\}$ on E).

Proposition 5.1. *If the normal frame Eq. (5.1) has solutions, then all frames $\{\tilde{X}_I\}$ normal on $U \subseteq E$ and adapted to local coordinates, defined on an open set $V \subseteq E$ such that $V \supseteq U$, are described by*

$$\tilde{X}_\mu|_U = (A^\nu_\mu X_\nu)|_U = \left.\frac{\partial}{\partial \tilde{u}^\mu}\right|_U, \qquad \tilde{X}_a|_U = (A^b_a X_b)|_U = \left.\frac{\partial}{\partial \tilde{u}^a}\right|_U, \tag{5.2}$$

where $\{X_I\}$ is the frame adapted to some arbitrarily fixed local coordinates $\{u^I\}$, defined on an open set containing or equal to V, $\{\tilde{u}^I\}$ are local coordinates with domain V and such that $\tilde{u}^a$ are solutions of (5.1), and $A^J_I = \partial u^J/\partial \tilde{u}^I$ on the intersection of the domains of $\{u^I\}$ and $\{\tilde{u}^I\}$.

Proof. Apply Proposition 4.1 for $e_I = \partial/\partial u^I$ and $\tilde{e}_I = \partial/\partial \tilde{u}^I$ and then use (3.2) and (3.3). □

This simple result gives a complete description of all normal frames, if any, adapted to (local) holonomic frames. It should be understood clearly, normal on U is the frame $\{\tilde{X}_I\}$, adapted to $\{\partial/\partial \tilde{u}^I\}$ and coinciding with it on U, but not the frame $\{\partial/\partial \tilde{u}^I\}$; in particular, the frame $\{\partial/\partial \tilde{u}^I\}$ is holonomic while the frame $\{\tilde{X}_I\}$ need not to be holonomic, even on U, if the connection considered does not satisfies some additional conditions, like the vanishment of its curvature on U.

Consider now briefly the existence problem for the solutions of (5.1). To begin with, we emphasize that in (5.1) enter only the fibre coordinates $\{\tilde{u}^a\}$, so that it leaves the basic ones $\{\tilde{u}^\mu\}$ completely arbitrary.

Proposition 5.2. *If E is of class C^2, $p \in E$ is fixed and $U = \{p\}$, then the general solution of (5.1) is*

$$\tilde{u}^a(q) = g^a + g^a_b\{-\Gamma^b_\mu(p)(q^\mu - p^\mu) + (q^b - p^b)\} + f^a_{IJ}(q)(q^I - p^I)(q^J - p^J), \tag{5.3}$$

where g^a and g^a_b are constants in $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\det[g^a_b] \neq 0, \infty$, the point q is in the domain V of $\{u^I\}$, $q^I := u^I(q)$, $p^I := u^I(p)$, and f^a_{IJ} are C^2 functions on V such that they and their first partial derivatives are bounded when $q^I \to p^I$.

Proof. Expand $\tilde{u}^a(q) = f^a(u^1(q), \ldots, u^n(q), \ldots, u^{n+r}(q)) = f^a(q^1, \ldots, q^{n+r})$ into a Taylor's first order polynomial with remainder term quadratic in $(q^I - p^I)$ and insert the result into (5.1). In this way one gets (5.3) with $g^a = \tilde{u}^a(p)$ and $g^a_b = (\partial \tilde{u}^a/\partial u^b)|_p$. □

Now we would like to investigate the existence of solutions of (5.1) along paths $\beta : J \to E$, i.e. for $U = \beta(J)$. The main result is formulated below as Proposition 5.3. For its proof, we shall need the following lemma.

Lemma 5.1. *Let $\gamma : J \to M$ be a regular C^1 injective path in a C^3 real manifold M. For every $s_0 \in J$, there exists a chart (U_1, x) of M such that $\gamma(s_0) \in U_1$, $x : U_1 \to J_1 \times \mathbb{R}^{\dim_{\mathbb{R}} M-1}$ for some open subinterval $J_1 \subseteq J$, $s_0 \in J_1$ and $x(\gamma(s)) = (s, \boldsymbol{t}_0)$ for all $s \in J_1$ and some fixed $\boldsymbol{t}_0 \in \mathbb{R}^{\dim_{\mathbb{R}} M-1}$.*

Proof. Let $s_0 \in J$ be a point in J which is not an end point of J, if any, and (U, y) be a chart with $\gamma(s_0)$ in its domain, $U \ni \gamma(s_0)$, and $y : U \to \mathbb{R}^{\dim_{\mathbb{R}} M}$. From the regularity of γ, $\dot\gamma \neq 0$, follows that at least one of the numbers $\dot\gamma_y^1(s_0), \dots, \dot\gamma_y^{\dim_{\mathbb{R}} M}(s_0)$, where $\gamma_y^i := y^i \circ \gamma$, is non-zero. We, without lost of generality, choose this non-vanishing component to be $\dot\gamma_y^1(s_0)$.[13] Then, due to the continuity of $\dot\gamma$ (γ is of class C^1) and according to the implicit function theorem [22, Chapter III, Section 8], [13, Sections 1.37 and 1.38], [23, Chapter 10, Section 2], there exists an *open* subinterval $J_1 \subseteq J$ containing s_0, $J_1 \ni s_0$, and such that $\dot\gamma^1|_{J_1} \neq 0$ and the restricted mapping $\gamma_y^1|_{J_1} : J_1 \to \gamma_y^1(J_1)$ is a C^1 diffeomorphism on its image. Define a neighborhood

$$U_1 := \{p | p \in U, y^1(p) \in \gamma_y^1(J_1)\} = y^{-1}(\gamma_y^1(J_1) \times \mathbb{R}^{\dim_{\mathbb{R}} M-1}) \ni \gamma(s_0),$$

and a chart (U_1, x) with local coordinate functions

$$x^1 := (\gamma_y^1|_{J_1})^{-1} \circ y^1, \qquad x^k := y^k - \gamma_y^k \circ x^1 + t_0^k, \quad k = 2, \dots, \dim_{\mathbb{R}} M, \tag{5.4}$$

where $t_0^k \in \mathbb{R}$ are arbitrarily fixed constant numbers. Since $\partial x^1/\partial y^j = (1/\dot\gamma_y^1)\delta_j^1$, $\partial x^k/\partial y^1 = -(\dot\gamma_y^k/\dot\gamma_y^1)$ for $k \geq 2$, and $\partial x^k/\partial y^l = \delta_l^k$ for $k, l \geq 2$, the Jacobian of the change $\{y^i\} \to \{x^i\}$ at $p \in U_1$ is $1/\dot\gamma^1(p) \neq 0, \infty$. Consequently $x : U_1 \to J_1 \times \mathbb{R}^{\dim_{\mathbb{R}} M-1}$ is really a coordinate homeomorphism with coordinate functions x^i.

In the new chart (U_1, x), the coordinates of $\gamma(s)$, $s \in J_1$ are

$$\gamma^1(s) := (x^1 \circ \gamma)(s) = s, \qquad \gamma^k(s) := (x^k \circ \gamma)(s) = t_0^k, \quad k \geq 2, \tag{5.5}$$

i.e. $x(\gamma(s)) = (s, \boldsymbol{t}_0)$ for some $\boldsymbol{t}_0 = (t_0^2, \dots, t_0^{\dim_{\mathbb{R}} M}) \in \mathbb{R}^{\dim_{\mathbb{R}} M-1}$. $\square$

Lemma 5.1 means that the chart (U_1, x) is so luckily chosen that the first coordinate in it of a point along γ coincides with the value of the corresponding path's parameter, the other coordinates being constant numbers. Moreover, in U_1 the path γ can be considered as a representative of a family of paths $\eta(\cdot, \boldsymbol{t}) : J_1 \to M$, $\boldsymbol{t} \in \mathbb{R}^{\dim_{\mathbb{R}} M-1}$, defined by $\eta(s, \boldsymbol{t}) := x^{-1}(s, \boldsymbol{t})$ for $(s, \boldsymbol{t}) \in J_1 \times \mathbb{R}^{\dim_{\mathbb{R}} M-1}$; indeed, $\gamma = \eta(\cdot, \boldsymbol{t}_0)$ or $\gamma(s) = \eta(s, \boldsymbol{t}_0)$, $s \in J_1 \subseteq J$.

Proposition 5.3. *Let Δ^h be a C^1 connection on a real C^3 bundle (E, π, M), $n = \dim M \geq 1$, and $r = \dim \pi^{-1}(x) \geq 1$ for $x \in M$. Let $\beta : J \to E$ be an injective regular C^1 path such that its tangent vector $\dot\beta(s)$ at s is not a vertical vector for all $s \in J$, $\dot\beta(s) \notin \Delta^v_{\beta(s)}$; in particular, the path β can be horizontal, i.e. $\dot\beta(s) \in \Delta^h_{\beta(s)}$ for all $s \in J$, but generally the vector $\dot\beta(s)$ can*

[13] If it happens that $\dot\gamma_y^1(s_0) = 0$ and $\dot\gamma_y^{i_0}(s_0) \neq 0$ for some $i_0 \neq 1$, we have simply to renumber the local coordinates to get $\dot\gamma_y^1(s_0) \neq 0$. Practically this is a transition to new coordinates $\{y^i\} \to \{z^i\}$ with $z^1 = y^{i_0}$ and, for instance, $z^{i_0} = y^1$ and $z^i = y^i$ for $i \neq 1, i_0$, in which the first component of $\dot\gamma$ is non-zero. We suppose that, if required, this coordinate change is already done. If occasionally it happens that $\dot\gamma_y^{j_0}(s) \neq 0$ for all $s \in J$ and fixed j_0, it is extremely convenient to take this particular component of $\dot\gamma$ as $\dot\gamma_y^1$—see the next sentence.

have also and a vertical component for some or all $s \in J$. Then, for every $s_0 \in J$, there exist a neighborhood U_1 of the point $\beta(s_0)$ in E and bundle coordinates $\{\tilde{u}^I\}$ on U_1 which are solutions of (5.1) for $U = U_1 \cap \beta(J) = \beta(J_1)$, with $J_1 := \{s \in J : \beta(s) \in U_1\}$, i.e. along the restricted path $\beta|_{J_1}$. All such bundle coordinates $\{\tilde{u}^I\}$ are given via Eq. (5.6).

Proof. Consider the chart (U_1, u) with $U_1 \ni \beta(s_0)$ provided by Lemma 5.1 for E and β instead of M and γ, respectively. For any $p \in U_1$, there is a unique $(s, \boldsymbol{t}) \in J_1 \times \mathbb{R}^{\dim_{\mathbb{R}} E-1}$ such that $p = u^{-1}(s, \boldsymbol{t})$, i.e., in the coordinates $\{u^I\}$ associated to u, the coordinates of p are $u^1(p) = s$ and $u^I(p) = t^I \in \mathbb{R}$ for $I \geq 2$. Besides, we have $u(\beta(s)) = (s, \boldsymbol{t_0})$ for all $s \in J_1$ and some fixed $\boldsymbol{t_0} \in \mathbb{R}^{\dim_{\mathbb{R}} E-1}$.

Since $\dot{\beta}(s)$ is not a vertical vector for all $s \in J$, the coordinates $\{u^I\}$ can be chosen to be *bundle* coordinates. For the purpose, in the proof of Lemma 5.1 one must choose $\{y^I\}$ as bundle coordinates and to take for $\dot{\beta}_y^1(s_0)$ any non-vanishing component between $\dot{\beta}_y^1(s_0), \ldots, \dot{\beta}_y^n(s_0)$, viz. if $\dot{\beta}_y^1(s_0) \neq 0$ the proof goes as it is written and, if $\dot{\beta}_y^1(s_0) = 0$, choose some μ_0 such that $\dot{\beta}_y^{\mu_0}(s_0) \neq 0$ and make, e.g., the change $\dot{\beta}_y^1(s_0) \leftrightarrow \dot{\beta}_y^{\mu_0}(s_0)$. This, together with (5.4), with u^I for x^k, ensures that $\{y^I\} \mapsto \{u^I\}$ is an admissible change, so that $\{u^I\}$ are bundle coordinates if the initial coordinates $\{y^I\}$ are such ones.

Let $\{u^I\}$ be so constructed bundle coordinates and $\eta := u^{-1}$, so that $\beta(s) = \eta(s, \boldsymbol{t}_0)$. Expanding $\tilde{u}^a(\eta(s, \boldsymbol{t}))$ into a first order Taylor's polynomial at the point $\boldsymbol{t}_0 \in K$, we find the general solution of (5.1), with $U = \beta(J_1) = U_1 \cap \beta(J)$, in the form

$$\begin{aligned}\tilde{u}^a(\eta(s, \boldsymbol{t})) = B^a(s) + B^a_b(s)\{&-\Gamma^b_\mu(\beta(s))[u^\mu(\eta(s, \boldsymbol{t})) - u^\mu(\beta(s))] \\ &+ [u^b(\eta(s, \boldsymbol{t})) - u^b(\beta(s))]\} + B^a_{IJ}(s, \boldsymbol{t}; \eta)[u^I(\eta(s, \boldsymbol{t})) \\ &- u^I(\beta(s))][u^J(\eta(s, \boldsymbol{t})) - u^J(\beta(s))],\end{aligned} \tag{5.6}$$

where $B^a, B^a_b : J_1 \to \mathbb{K} = \mathbb{R}$, $\det[B^a_b] \neq 0, \infty$, and the C^1 functions B^a_{IJ} and their first partial derivatives are bounded when $\boldsymbol{t} \to \boldsymbol{t}_0$. (Notice, the terms with $\mu = 1$ and/or $I = 1$ and/or $J = 1$ do not contribute in (5.6) as $u^1(\eta(s, \boldsymbol{t})) \equiv s$ and, besides, the functions B^a_{IJ} can be taken symmetric in I and J, $B^a_{IJ} = B^a_{JI}$.) □

Remark 5.1. If there is $s_0 \in J$ for which $\dot{\beta}(s_0)$ is a vertical vector, $\dot{\beta}(s_0) \in \Delta^v_{\beta(s_0)}$, then Proposition 5.3 remains true with the only correction that the coordinates $\{u^I\}$ will *not be bundle* coordinates. If this is the case, the constructed coordinates $\{\tilde{u}^I\}$ will be solutions of (5.1), but we cannot assert that they are bundle coordinates which are (locally) normal along β in a neighborhood of the point $\beta(s_0)$.

Proposition 5.3 can be generalized by requiring β to be locally injective instead of injective, i.e. for each $s \in J$ to exist a subinterval $J_s \subseteq J$ such that $J_s \ni s$ and the restricted path $\beta|_{J_s}$ to be injective. Besides, if one needs a version of the above results for complex bundles, they should be considered as real ones (with doubled dimension of the manifolds) for which are applicable the above considerations.

Corollary 5.1. *At any arbitrarily fixed point in E and/or along a given injective regular C^1 path in E, whose tangent vector is not vertical, there exist (possibly local, in the latter case) normal frames.*

Proof. See Definition 4.1, Propositions 5.2 and 5.3, and Eq. (5.1). If the path is not contained in a single coordinate neighborhood, one should cover its image in the bundle space with such neighborhoods and, then, to apply Proposition 5.3; in the intersection of the coordinate domains, the uniqueness (and, possibly, continuity or differentiability) of the normal frames may be lost. □

Definition 5.1. Local bundle coordinates $\{\tilde{u}^I\}$, defined on an open set $V \subseteq E$, will be called *normal* on $U \subseteq V$ for a connection Δ^h if the frame $\{\tilde{X}_I\}$ in $T(E)$ adapted to $\{\partial/\partial\tilde{u}^I\}$ over V is normal for Δ^h on U.

Corollary 5.1 implies the existence of coordinates normal at a given point or (locally) along a given injective path whose tangent vector is not vertical; in particular, there exist coordinates normal along an injective horizontal path. However, normal coordinates generally do not exist on more general subsets of the bundle space E. A criterion for existence of coordinates normal on sufficiently general subsets $U \subseteq E$, e.g. on 'horizontal' submanifolds, is given by Theorem A.1 in Appendix A. In particular, we have the following corollary from this theorem.

Proposition 5.4. *If Δ^h is a C^1 connection, U is an open set in E, and normal frames for Δ^h on U exist, then there are holonomic such frames if Δ^h is flat on U. Said otherwise, the system of Eq.* (5.1) *may admit solutions on an open set U if*

$$R^a_{\mu\nu}|_U = 0, \tag{5.7}$$

where

$$R^a_{\mu\nu} = \partial_\mu(\Gamma^a_\nu) - \partial_\nu(\Gamma^a_\mu) + \Gamma^b_\mu\partial_b(\Gamma^a_\nu) - \Gamma^b_\nu\partial_b(\Gamma^a_\mu) = X_\mu(\Gamma^a_\nu) - X_\nu(\Gamma^a_\mu) \tag{5.8}$$

are the (*fibre*) *components of the curvature of Δ^h in some frame $\{X_I\}$ on E adapted to a holonomic one* (*see also* [19]).

Remark 5.2. However, in the general case the flatness of a connection on an open set is only a necessary, but not sufficient, condition for the existence of coordinates normal on that set—see Theorem A.1 in Appendix A. Exceptions are the linear connections on vector bundles—see Remark A.3 in Appendix A. One can easily show that part of the integrability conditions for (5.1) for an open set U are

$$0 = \frac{\partial^2\tilde{u}^a}{\partial u^\nu\partial u^\mu} - \frac{\partial^2\tilde{u}^a}{\partial u^\mu\partial u^\nu} \equiv \frac{\partial\tilde{u}^a}{\partial u^b}R^b_{\mu\nu} \tag{5.9}$$

from where Proposition 5.4 immediately follows. However, the flatness of the connection on U generally does not imply the rest of the integrability conditions, viz. $\partial^2\tilde{u}^a/\partial u^b\partial u^\mu - \partial^2\tilde{u}^a/\partial u^\mu\partial u^b = 0$ and $\partial^2\tilde{u}^a/\partial u^b\partial u^c - \partial^2\tilde{u}^a/\partial u^c\partial u^b = 0$.

The combination of Propositions 5.4 and 4.2 implies the non-existence of coordinates normal on an open set for non-flat (non-integrable) connections.

6. Normal frames on vector bundles

The normal frames for covariant derivative operators (linear connections), other derivations, and linear transports along paths are known and studied objects in vector bundles [10,11]. The goal of the present section is to be made a link between them and the general theory of Section 4.

Consider a linear connection Δ^h on a vector bundle (E, π, M), i.e. a connection the assigned to which parallel transport is a linear mapping. Let the frame $\{e_I\}$ in $T(E)$ be given by (3.33) and $\{X_I\}$ be the frame adapted to $\{e_I\}$ for Δ^h. Then, by Proposition 3.1, the 2- and 3-index coefficients of Δ^h are connected via (3.34) in which $\{u^a\}$ are vector fibre coordinates.

Proposition 6.1. *A frame $\{X_I\}$ is normal on $U \subseteq E$ for a linear connection Δ^h if and only if in it vanish the 3-index coefficients of Δ^h on $\pi(U) \subseteq M$,*

$$\Gamma^a_\mu|_U = 0 \iff \Gamma^a_{b\mu}|_{\pi(U)} = 0. \tag{6.1}$$

Proof. Since $u^{n+1}, \dots, u^{n+r}$ are 1-forms which are linearly independent for all $p \in U$, the assertion follows from Eq. (3.34). □

Combining Proposition 6.1 with (3.43), we see that the normal frame Eq. (4.1) in vector bundle is equivalent to

$$(B^\nu_\mu \Gamma^a_{b\nu} + B^a_{b\mu})|_{\pi(U)} = 0 \tag{6.2}$$

or to its matrix variant (see also (3.43′); $\Gamma_\nu := [\Gamma^a_{b\nu}]$, $B_\mu := [B^a_{b\mu}]$)

$$(B^\nu_\mu \Gamma_\nu + B_\mu)|_{\pi(U)} = 0. \tag{6.2′}$$

Taking into account (6.2) and (3.42), we can assert that the frame $\{\tilde{X}_I\}$ adapted to the frame

$$(\tilde{e}_\mu, \tilde{e}_a) = (e_\nu, e_b) \cdot \begin{bmatrix} B^\nu_\mu \circ \pi & 0 \\ -((B^\lambda_\mu \Gamma^b_{c\lambda}) \circ \pi) \cdot u^c & B^b_a \circ \pi \end{bmatrix}, \tag{6.3}$$

where $[B^\nu_\mu]$ and $[B^a_b]$ are non-degenerate matrix-valued functions, is normal on U for Δ^h and hence $\tilde{X}_I = \tilde{e}_I$, by virtue of Proposition 4.2. Recall (see (3.21), (3.22), and (3.29)), the change $\{e_I\} \mapsto \{\tilde{e}_I\}$, given by (6.3), entails $\{X_I\} \mapsto \{\tilde{X}_I\}$, where

$$\tilde{X}_\mu = (B^\nu_\mu \circ \pi) X_\nu, \qquad \tilde{X}_a = (B^b_a \circ \pi) X_b, \tag{6.4}$$

which is equivalent to $\{E_I\} \mapsto \{\tilde{E}_I\}$ with

$$\tilde{E}_\mu = B^\nu_\mu E_\nu, \qquad \tilde{E}_a = B^b_a E_b. \tag{6.5}$$

Here (see (3.27)) $\{E_\mu = \pi_*|_{\Delta^h}(X_\mu)\}$ is a frame in $T(M)$ and $\{E_a = v^{-1}(X_a)\}$ is a frame in E.

Thus, if additional restriction are not imposed, the theory of normal frames in vector bundles is rather trivial, which reflects a similar situation in general bundles, considered in Section 4. However, the really interesting and sensible case is when one considers frames compatible with the covariant derivatives. As we know (see (3.44)), it corresponds to arbitrary non-degenerate matrix-valued functions $[B^\nu_\mu]$ and $B = [B^a_b]$ and a matrix-valued functions $B_\mu = [B^\nu_{b\mu}]$ given by

$$B_\mu = \tilde{E}_\mu(B) \cdot B^{-1} = B^\nu_\mu E_\nu(B) \cdot B^{-1}. \tag{6.6}$$

In particular, such are all holonomic frames in $T(E)$, locally induced by local coordinates on E. Now the normal frames Eq. (6.2) (or (4.1)) reduces to

$$(\Gamma_\mu \cdot B + E_\mu(B))|_{\pi(U)} = 0. \tag{6.7}$$

This equation leaves the frame $\{\tilde{E}_\mu = \pi_*|_{\Delta^h}(X_\mu)\}$ in $T(M)$ completely arbitrary and imposes restriction on the frame $\{\tilde{E}_a = v^{-1}(X_a) = B^b_a E_b\}$ in E. This conclusion justifies the following definition.

Definition 6.1. Given a *linear* connection Δ^h on a *vector* bundle (E, π, M) and a subset $U_M \subseteq M$. A *frame* $\{E_a\}$ *in* E, defined over an open set V_M containing U_M or equal to it, $V_M \supseteq U_M$, is called *normal for* Δ^h *over/on* U_M if their is a frame $\{X_I\}$ in $T(E)$, defined over an open set $V_E \subseteq E$, which is normal for Δ^h over a subset $U_E \subseteq E$ and such that $\pi(U_E) = U_M$, $\pi(V_E) = V_M$, and $E_a = v^{-1}(X_a)$, with the mapping v defined by (3.14). Respectively, $\{E_a\}$ is normal for Δ^h along a mapping $g : Q_M \to M$, $Q_M \neq \emptyset$, if $\{E_a\}$ is normal for Δ^h over $g(Q_M)$.

Taking into account Definition 4.1, we see that the so-defined normal frames in the bundle space E are just the ones used in the theory of frames normal for linear connections in vector bundles [10,11,7–9].

It is quite clear, to any frame $\{X_I\}$ in $T(E)$ normal over $U \subseteq E$, there corresponds a unique frame $\{E_a = v^{-1}(X_a)\}$ in E normal over $\pi(U) \subseteq M$. But, to a frame $\{E_a\}$ in E normal over $\pi(U)$, there correspond infinitely many frames $\{X_I\} = \{(\pi_*|_{\Delta^h})^{-1}(E_\mu), v(E_a)\}$ in $T(E)$ normal over U, where $\{E_\mu\}$ is an arbitrary frame in $T(M)$ over $\pi(U)$. Thus the problems of existence and (un)uniqueness of normal frames in $T(E)$ is completely reduced to the same problems for normal frames in E. The last kind of problems, as we noted at the beginning of the present section, are known and investigated and the reader is referred to [10,11,7–9] for their solutions and further details.

We emphasize that a normal frame $\{E_a\}$ in E, as well as the basis $\{v(E_a)\}$ for Δ^v, can be holonomic as well as anholonomic (see *loc. cit.*); at the same time, a normal frame $\{X_I\}$ in $T(E)$ is anholonomic unless some conditions hold, a necessary condition being the flatness (integrability) of the horizontal distribution Δ^h.

Ending this section, let us say some words regarding frames normal for affine connections on vector bundles.

Proposition 6.2 (cf. Proposition 6.1). *A frame $\{X_I\}$ is normal on $U \subseteq E$ for an affine connection Δ^h, with 2-index coefficients (3.54) on U, if and only if in it is fulfilled*

$$\Gamma^a_{b\mu}|_{\pi(U)} = 0, \tag{6.8a}$$

$$G^a_{\mu}|_{\pi(U)} = 0. \tag{6.8b}$$

Proof. The assertion follows from Definition 4.1, Eq. (3.54), and the linear independence of the vector fibre coordinates $u^{n+1}, \ldots, u^{n+r}$, considered as 1-forms. □

Corollary 6.1. *A necessary condition for existence of frames normal on $U \subseteq E$ for an affine connection is*

$$G^a_{\mu}|_{\pi(U)} = 0 \tag{6.9}$$

in all adapted frames on U.

Proof. Use (6.8b) and (3.56b). □

Corollary 6.2. *A necessary condition for existence of frames normal on $U \subseteq E$ for an affine connection is*

$$\Gamma^a_{\mu}|_U = -\{(\Gamma^a_{b\mu} \circ \pi) \cdot u^b\}|_U \tag{6.10}$$

in all adapted frames on U; in particular, if U is an open set, then (6.10) means that the restriction of the connection considered on $(U, \pi|_U, \pi(U))$ is a linear connection.

Proof. Apply Proposition 6.2 and Corollary 6.1. □

Corollary 6.3. *If an affine connection admits frames normal on $U \subseteq E$, then all of them are normal on U for the linear connection, corresponding to it via Eq. (3.54), and vice versa.*

Proof. Use Corollary 6.2 and Proposition 6.1. □

Thus, if the condition (6.9) is satisfied, the above results completely reduce the problems of existence, (un)uniqueness and the properties of frames normal for affine connections to the same problems for linear connections (that correspond to them).

7. Conclusion

In Section 4, we saw that the theory of normal frames in the most general case is quite trivial. This reflects the understanding that the more general a concept is, the less particular properties it has, but the more concrete applications it can find if it is restricted somehow. This situation was demonstrate when normal frames adapted to holonomic ones were considered; e.g. they exist at a given point or along an injective horizontal path, but on an open set they may exist only in the flat case. A feature of a vector bundle (E, π, M) is that the frames in $T(E)$ over E are in bijective correspondence with pairs of frames in E over M and in $T(M)$

over M. This result allows the normal frames in $T(E)$, if any, to be 'lowered' to ones in E. From here a conclusion was made that the theory of frames in $T(E)$ normal for linear connections on a vector bundle is equivalent to the existing one of frames in E normal for covariant derivatives in (E, π, M) [10,11].

It should be emphasized, the importance of the normal frames for the physics comes from the fact that they are the mathematical object corresponding to the physical concept of inertial frame of reference [24,10,25].

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Appendix A. Coordinates normal along injective mappings with non-vanishing horizontal component

The purpose of this appendix is a multi-dimensional generalization of Proposition 5.3 in the real case, $\mathbb{K} = \mathbb{R}$. It is formulated below as Theorem A.1. For its proof we shall need a result which is a multidimensional generalization of Lemma 5.1.

Lemma A.1. *Let $n \in \mathbb{N}$, M be a C^3 manifold with $\dim M \geq n$, J^n be an open set in $\mathbb{R}^n$, and $\gamma : J^n \to M$ be C^1 regular injective mapping. For every $s_0 \in J^n$, there exists a chart (U_1, x) of M such that $\gamma(s_0) \in U_1$, $x : U_1 \to J_1^n \times \mathbb{R}^{\dim_{\mathbb{R}} M - n}$ for some open subset $J_1^n \subseteq J^n$, $s_0 \in J_1^n$, and $x(\gamma(s)) = (s, \boldsymbol{t}_0)$ for some fixed $\boldsymbol{t_0} \in \mathbb{R}^{\dim_{\mathbb{R}} M - n}$ and all $s \in J_1^n$.*

Proof. Let us choose arbitrary some $s_0 \in J^n$ and a chart (U, y) with $U \ni \gamma(s_0)$ and $y : U \to \mathbb{R}^{\dim_{\mathbb{R}} M}$. Since the regularity of γ at s_0 means that $[(\partial \gamma_y^i / \partial s^a)|_{s_0}]$ has maximal rank, equal to n, we, without loss of generality, can suppose the coordinates $\{y^i\}$ to be taken such that $\det[(\partial \gamma_y^a / \partial s^b)|_{s_0}] \neq 0, \infty$.[14] Then the implicit function theorem [13,22,23] implies the existence of a subneighborhood $J_1^n \subseteq J^n$ with $J_1^n \ni s_0$ and such that the matrix $[(\partial \gamma_y^a / \partial s^b)|_s]$ is non-degenerate for $s \in J_1^n$ and the mapping

$$(\gamma_y^1, \dots, \gamma_y^n)|_{J_1^n} : J_1^n \to (\gamma_y^1(J_1^n), \dots, \gamma_y^n(J_1^n)) \subseteq \mathbb{R}^n,$$

with $(\gamma_y^1, \dots, \gamma_y^n)|_{J_1^n} : s \mapsto (\gamma_y^1(s), \dots, \gamma_y^n(s))$ for $s \in J_1^n$, is a C^1 diffeomorphism. Define a chart (U_1, x) of M with domain

$$\begin{aligned} U_1 &:= \{p | p \in U, y^a(p) \in \gamma_y^a(J_1^n), a = 1, \dots, n\} \\ &= y^{-1}((\gamma_y^1(J_1^n), \dots, \gamma_y^n(J_1^n)) \times \mathbb{R}^{\dim_{\mathbb{R}} M - n}) \ni \gamma(s_0) \end{aligned} \tag{A.1a}$$

[14] If we start from a chart (U, z) for which the matrix $[(\partial \gamma_z^a / \partial s^b)|_{s_0}]$ is degenerate, we can make a coordinate change $\{z^i\} \to \{y^i\}$ with $y^i = z^{\alpha_i}$, where the integers $\alpha_1, \dots, \alpha_{\dim_{\mathbb{R}} M}$ form a permutation of $1, \dots, \dim_{\mathbb{R}} M$, such that $[(\partial \gamma_y^a / \partial s^b)|_{s_0}]$ is non-degenerate. (For the proof, see any book on matrices, e.g. [26,27].) Further, we suppose that such a renumbering of the local coordinates is already done if required. (cf. Footnote 13).

and local coordinate functions x^i given via

$$y^a =: (\gamma_y^a\big|_{J_1^n}) \circ (x^1, \dots, x^n), \quad a = 1, \dots, n,$$
$$y^k =: x^k + (\gamma_y^k\big|_{J_1^n}) \circ (x^1, \dots, x^n) - t_0^k, \quad k = n+1, \dots, \dim_{\mathbb{R}} M, \tag{A.1b}$$

where $(x^1, \dots, x^n) : p \mapsto (x^1(p), \dots, x^n(p))$, $p \in U_1$, and $t_0^k \in \mathbb{R}$ are constant numbers. Since $\partial y^a/\partial x^b = \partial\gamma_y^a/\partial s^b$, $\partial y^a/\partial x^k = \delta_k^a$ for $k \geq n+1$, $\partial y^k/\partial x^a = \partial\gamma_y^k/\partial s^a$ for $k \geq n+1$, and $\partial y^k/\partial x^l = \delta_l^k$ for $k, l \geq n+1$, the Jacobian of the change $\{y^i\} \to \{x^i\}$ on U_1 is $\det[\partial x^i/\partial y^j] = (\det[\partial\gamma_y^a/\partial s^b])^{-1} \neq 0, \infty$. Consequently x^i are really coordinate functions and $x : U_1 \to J_1^n \times \mathbb{R}^{\dim_{\mathbb{R}} M - n}$ is in fact coordinate homeomorphism.[15] The coordinates $\{x^i\}$ can be expressed through $\{y^i\}$ explicitly. Indeed, writing the first raw of (A.1b) as

$$(y^1, \dots, y^n) = (\gamma_y^1|_{J_1^n}, \dots, \gamma_y^n|_{J_1^n}) \circ (x^1, \dots, x^n) = (\gamma_y^1, \dots, \gamma_y^n)|_{J_1^n} \circ (x^1, \dots, x^n),$$

and using that $(\gamma_y^1, \dots, \gamma_y^n)|_{J_1^n}$ is a C^1 diffeomorphism and the second raw of (A.1b), we find (cf. (5.4))

$$(x^1, \dots, x^n) = ((\gamma_y^1, \dots, \gamma_y^n)|_{J_1^n})^{-1} \circ (y^1, \dots, y^n),$$
$$x^k = y^k - (\gamma_y^k|_{J_1^n}) \circ ((\gamma_y^1, \dots, \gamma_y^n)|_{J_1^n})^{-1} \circ (y^1, \dots, y^n) + t_0^k, \quad k \geq n+1. \tag{A.1b$'$}$$

Using (A.1), we see that in (U_1, x) the local coordinates of $\gamma(s)$ for $s = (s^1, \dots, s^n) \in J_1^n$ are

$$\gamma^a(s) := x^a(\gamma(s)) = s^a, \qquad \gamma^k(s) := x^k(\gamma(s)) = t_0^k, \quad k \geq n+1, \tag{A.2}$$

i.e. $x(\gamma(s)) = (s, \boldsymbol{t}_0)$ for some fixed $\boldsymbol{t}_0 = (t_0^{n+1}, \dots, t_0^{\dim_{\mathbb{R}} M}) \in \mathbb{R}^{\dim_{\mathbb{R}} M - n}$. $\square$

Thus, in the chart (U_1, x) or the coordinates $\{x^i\}$ constructed above, the first n coordinates of a point lying in $\gamma(J^n)$, i.e. in $\gamma(J_1^n)$, coincide with the corresponding parameters $s^1, \dots, s^n$ of γ, the remaining coordinates, if any, being constant numbers. This conclusion allows locally, in U_1, the mapping γ to be considered as a representative of a family of mappings $\eta(\cdot, \boldsymbol{t}) : J_1^n \to M$, $\boldsymbol{t} \in \mathbb{R}^{\dim_{\mathbb{R}} M - n}$, defined by $\eta(s, \boldsymbol{t}) := x^{-1}(s, \boldsymbol{t})$ for $(s, \boldsymbol{t}) \in J_1^n \times \mathbb{R}^{\dim_{\mathbb{R}} M - n}$. In fact, we have $\gamma = \eta(\cdot, \boldsymbol{t}_0)$ or $\gamma(s) = \eta(s, \boldsymbol{t}_0)$.[16]

Let (E, π, M) be a C^3 bundle endowed with C^1 connection Δ^h. Let $k \in \mathbb{N}$, $k \leq \dim M$, and J^k be an open set in $\mathbb{R}^k$. Consider a C^2 regular injective mapping $\beta : J^k \to E$ such that the vector fields $\dot\beta_\alpha : s \mapsto \dot\beta_\alpha(s) := (\partial\beta^I(s)/\partial s^\alpha)(\partial/\partial u^I)|_{\beta(s)}$, with $s := (s^1, \dots, s^k) \in J^k$ and $\alpha = 1, \dots, k$, do *not* belong to the vertical distribution Δ^v, $\dot\beta_\alpha(s) \notin \Delta^v_{\beta(s)}$ for all $s \in J^k$; in particular, the mapping β can be a horizontal mapping in a sense that $\dot\beta_\alpha(s) \in \Delta^h_{\beta(s)}$ for all $s \in J^k$, but generally these vectors can have a vertical component too. Our aim is to

[15] The so-constructed chart (U_1, x) is, obviously, a multidimensional generalization of a similar chart defined in the proof of Lemma 5.1—see the paragraph containing Eq. (5.4).

[16] In [9] the existence of η is taken as a given fact without proof.

find the integrability conditions for the normal frame/coordinates Eq. (5.1) and its solutions, if any, when $U = \beta(J_1^k)$ for some subset $J_1^k \subseteq J^k$.

Let us take some $s_0 \in J^k$ and construct the chart (U_1, u) with $U_1 \ni \beta(s_0)$ provided by Lemma A.1 with E for M and β for γ. If $J_1^k := \{s \in J^k : \beta(s) \in U_1\}$ and $p \in U_1$, then there is a unique $(s, \boldsymbol{t}) \in J_1^k \times \mathbb{R}^{\dim_{\mathbb{R}} E-k}$ such that $p = \eta(s, \boldsymbol{t})$ with $\eta := u^{-1}$, i.e. $u^I(p) = s^I$ for $I = 1, \dots, k$ and $u^I(p) = t^I$ for $I = k+1, \dots, n+r$. Besides, we have $u(\beta(s)) = (s, \boldsymbol{t}_0)$ for all $s \in J_1^k$ and some fixed $\boldsymbol{t}_0 \in \mathbb{R}^{\dim_{\mathbb{R}} E-k}$. Since the vector fields $\dot{\beta}_\alpha$, $\alpha = 1, \dots, k$, are not vertical, we can construct the coordinates $\{u^I\}$, associated to the chart (U_1, u), so that they to be *bundle* coordinates on U_1 (see the proof of Lemma A.1). Thus on U_1 we have bundle coordinates $\{u^I\}$ such that

$$\begin{aligned}
&(u^1(\eta(s,\boldsymbol{t})), \dots, u^{n+r}(\eta(s,\boldsymbol{t}))) := (s,\boldsymbol{t}) \in \mathbb{R}^{n+r},\\
&s = (s^1, \dots, s^k) \in J_1^k,\ \boldsymbol{t} = (t^{k+1}, \dots, t^{n+r}) \in \mathbb{R}^{n+r-k}.
\end{aligned} \tag{A.3}$$

Let the indices α and β run from 1 to k and the indices σ and τ take the values form $k+1$ to n; we set $\sigma = \tau = \emptyset$ if $k = n$. Thus, we have $u^\alpha(\eta(s,\boldsymbol{t})) = s^\alpha$, $u^\sigma(\eta(s,\boldsymbol{t})) = t^\sigma$, and $u^a(\eta(s,\boldsymbol{t})) = t^a$.

Proposition A.1. *Under the hypotheses made above, the normal frame/coordinates Eq.* (5.1) *with* $U = \beta(J_1^k) = \beta(J^k) \cap U_1$ *has solutions if and only if the system of equations*

$$\left.\left(\frac{\partial \Gamma_\alpha^b}{\partial u^\beta} - \frac{\partial \Gamma_\beta^b}{\partial u^\alpha}\right)\right|_{\beta(s)} B_b^a(s) + \Gamma_\alpha^b(\beta(s))\frac{\partial B_b^a(s)}{\partial s^\beta} - \Gamma_\beta^b(\beta(s))\frac{\partial B_b^a(s)}{\partial s^\alpha} = 0, \tag{A.4}$$

where Γ_μ^a *are the 2-index coefficients of* Δ^h *in* $\{u^I\}$, *has solutions* $B_b^a : J_1^k \to \mathbb{R}$ *with* $\det[B_b^a] \neq 0, \infty$. *Besides, if such solutions exist, then all solutions of* (5.1) *are given on* U_1 *by the formula*

$$\begin{aligned}
\tilde{u}^a(\eta(s,\boldsymbol{t})) = &-\int_{s_1}^{s} B_b^a(s)\Gamma_\alpha^b(\beta(s))\,\mathrm{d}s^\alpha - B_b^a(s)\Gamma_\mu^b(\beta(s))[u^\mu(\eta(s,\boldsymbol{t})) - u^\mu(\beta(s))]\\
&+ B_b^a(s)[u^b(\eta(s,\boldsymbol{t})) - u^b(\beta(s))] + f_{\mu\nu}^a(s;\boldsymbol{t};\eta)[u^\mu(\eta(s,\boldsymbol{t}))\\
&- u^\mu(\beta(s))][u^\nu(\eta(s,\boldsymbol{t})) - u^\nu(\beta(s))],
\end{aligned} \tag{A.5}$$

where $s_1 \in J_1^k$ *is arbitrarily fixed,* B_b^a, *with* $\det[B_b^a] \neq 0, \infty$, *are solutions of* (A.4), *and the functions* $f_{\mu\nu}^a$ *and their first partial derivatives are bounded when* $\boldsymbol{t} \to \boldsymbol{t}_0$.

Remark A.1. As $u^\alpha(\eta(s,\boldsymbol{t})) = u^\alpha(\beta(s)) \equiv s^\alpha$ for all $\alpha = 1, \dots, k$, the terms with $\mu, \nu = 1, \dots, k$ in (A.5) have vanishing contribution.

Remark A.2. For $k = 1$, we have $\alpha = \beta = 1$, due to which Eqs. (A.4) are identically valid and Proposition A.1 reduces to Proposition 5.3.

Proof. To begin with, we rewrite (5.1) as

$$\left.\frac{\partial \tilde{u}^a}{\partial s^\alpha}\right|_{\beta(s)} = -\left.\frac{\partial \tilde{u}^a}{\partial t^b}\right|_{\beta(s)} \Gamma_\alpha^b(\beta(s)), \qquad \left.\frac{\partial \tilde{u}^a}{\partial t^\sigma}\right|_{\beta(s)} = -\left.\frac{\partial \tilde{u}^a}{\partial t^b}\right|_{\beta(s)} \Gamma_\sigma^b(\beta(s)).$$

Introducing a non-degenerate matrix-valued function $[B^a_b]$ on J^k_1 by

$$B^a_b(s) = \left.\frac{\partial \tilde{u}^a}{\partial t^b}\right|_{\beta(s)} = \left.\frac{\partial \tilde{u}^a(s, \boldsymbol{t})}{\partial t^b}\right|_{\boldsymbol{t}=\boldsymbol{t}_0}, \tag{A.6}$$

we see that (5.1) is equivalent to

$$\left.\frac{\partial \tilde{u}^a}{\partial s^\alpha}\right|_{\beta(s)} = -B^a_b(s)\Gamma^b_\alpha(\beta(s)), \quad \alpha = 1, \dots, k, \tag{A.7a}$$

$$\left.\frac{\partial \tilde{u}^a}{\partial t^\sigma}\right|_{\beta(s)} = -B^a_b(s)\Gamma^b_\sigma(\beta(s)), \quad \sigma = k+1, \dots, n. \tag{A.7b}$$

Expanding $\tilde{u}^a(\eta(s, \boldsymbol{t}))$ into a Taylor's polynomial up to second order terms relative to $(\boldsymbol{t} - \boldsymbol{t}_0)$ about the point $\boldsymbol{t}_0$ and using (A.6) and (A.7), we get:

$$\begin{aligned}\tilde{u}^a(\eta(s, \boldsymbol{t})) &= f^a(s) - B^a_b(s)\Gamma^b_\sigma(\beta(s))[t^\sigma - t^\sigma_0] + B^a_b(s)[t^b - t^b_0] \\ &\quad + f^a_{\sigma\tau}(s; \boldsymbol{t}; \eta)[t^\sigma - t^\sigma_0][t^\tau - t^\tau_0] \\ &= f^a(s) - B^a_b(s)\Gamma^b_\mu(\beta(s))[u^\mu(\eta(s, \boldsymbol{t})) - u^\mu(\beta(s))] \\ &\quad + B^a_b(s)[u^b(\eta(s, \boldsymbol{t})) - u^b(\beta(s))] \\ &\quad + f^a_{\mu\nu}(s; \boldsymbol{t}; \eta)[u^\mu(\eta(s, \boldsymbol{t})) - u^\mu(\beta(s))][u^\nu(\eta(s, \boldsymbol{t})) - u^\nu(\beta(s))],\end{aligned} \tag{A.8}$$

where f^a and $f^a_{\mu\nu}$ are C^1 functions and $f^a_{\mu\nu}$ and their first partial derivatives are bounded when $\boldsymbol{t} \to \boldsymbol{t}_0$. Eq. (A.7a) is the only condition that puts some restrictions on f^a and B^a_b (besides $\det[B^a_b] \neq 0, \infty$). Inserting (A.8) into (A.7a) and using that $\beta(s) = \eta(s, \boldsymbol{t}_0)$, we obtain

$$\frac{\partial f^a(s)}{\partial s^\alpha} = -B^a_b(s)\Gamma^b_\alpha(\beta(s)). \tag{A.9}$$

Thus the initial normal coordinates equation (5.1), with $U = \beta(J^k_1)$, has solutions if and only if there exist solutions of (A.9) relative to f^a and/or B^a_b. The integrability conditions for (A.9) are [20]

$$0 = \frac{\partial^2 f^a}{\partial s^\beta \partial s^\alpha} - \frac{\partial^2 f^a}{\partial s^\alpha \partial s^\beta} = -\frac{\partial}{\partial s^\beta}(B^A_b(s)\Gamma^b_\alpha(s)) + \frac{\partial}{\partial s^\alpha}(B^A_b(s)\Gamma^b_\beta(s)) = \cdots,$$

and coincide with (A.4), by virtue of $u^\alpha(\beta(s)) = s^\alpha$. This result concludes the proof of the fires part of the proposition.

If (A.4) admits solutions B^a_b with $\det[B^a_b] \neq 0, \infty$, then the general solution of (A.9) is $f^a(s) = -\int_{s_1}^{s} B^a_b(s)\Gamma^b_\alpha(\beta(s))\,\mathrm{d}s^\alpha$ for some $s_1 \in J^k_1$ and this solution is independent of the integration path in J^k_1, due to (A.4). □

Lemma A.2. *Let (E, π, M) be a C^3 bundle endowed with C^2 connection with coefficients Γ^a_μ in the frame adapted to local coordinates $\{u^i\}$, defined before Proposition A.1. There* *exist solutions* B^a_b *with* $\det[B^a_b] \neq 0, \infty$ *of the system of Eq.* (A.4) *if and only if the coefficients* Γ^a_μ *satisfy the equations*

$$R^a_{\alpha\beta}(\beta(s)) = 0, \quad s \in J^k_1, \tag{A.10a}$$

$$\left.\left(\Gamma^d_\alpha \frac{\partial^2 \Gamma^c_\beta}{\partial u^b \partial u^d} - \Gamma^d_\beta \frac{\partial^2 \Gamma^c_\alpha}{\partial u^b \partial u^d} \right)\right|_{\beta(s)} = 0, \quad s \in J^k_1, \tag{A.10b}$$

in which $R^a_{\mu\nu}$ *are the* (*fibre*) *components in* $\{u^I\}$ *of the curvature of* Δ^h, *defined by* (5.8). *If the conditions* (A.10) *are valid, the set of the solutions of* (A.4) *coincides with the set of solutions of the system*

$$\frac{\partial B^a_b(s)}{\partial s^\alpha} = -B^a_c(s) \left.\frac{\partial \Gamma^c_\alpha}{\partial u^b}\right|_{\beta(s)} + \frac{\partial D^a_b(s)}{\partial s^\alpha} \tag{A.11}$$

relative to B^a_b, *where* D^a_b *are solutions of*

$$\left(\Gamma^b_\alpha(\beta(s)) \frac{\partial}{\partial s^\beta} - \Gamma^b_\beta(\beta(s)) \frac{\partial}{\partial s^\alpha} \right) D^a_b(s) = 0. \tag{A.12}$$

Proof. Consider the integrability condition (A.4) for (5.1) in more details. Define functions $D^a_{b\alpha} : J^k_1 \to \mathbb{K} = \mathbb{R}$ via the equation

$$\frac{\partial B^a_b(s)}{\partial s^\alpha} = -B^a_c(s) \left.\frac{\partial \Gamma^c_\alpha}{\partial u^b}\right|_{\beta(s)} + D^a_{b\alpha}(s). \tag{A.13}$$

The substitution of this equality into (A.4) results in

$$R^b_{\beta\alpha}(\beta(s)) B^a_b(s) - \Gamma^b_\alpha(\beta(s)) D^a_{b\beta}(s) + \Gamma^b_\beta(\beta(s)) D^a_{b\alpha}(s) = 0,$$

where $R^a_{\alpha\beta}$ are the (fibre) components in $\{u^I\}$ of the curvature of Δ^h, defined by (5.8). The simple observation that $\{\tilde{u}^\alpha, \tilde{u}^a\}$, if they exist as solutions of (5.1), are normal coordinates on the whole bundle space of the restricted bundle $(U, \pi|_U, \pi(U))$ with $U = \beta(J^k_1)$ leads to

$$R^a_{\alpha\beta}(\beta(s)) = 0, \quad s \in J^k_1, \tag{A.14}$$

by virtue of Proposition 5.4. Therefore the previous equation reduces to

$$\Gamma^b_\alpha(\beta(s)) D^a_{b\beta}(s) - \Gamma^b_\beta(\beta(s)) D^a_{b\alpha}(s) = 0. \tag{A.15a}$$

It is clear that (A.13)–(A.15a) are equivalent to (A.4). Consequently, the quantities $D^a_{b\alpha}$ must be solutions of (A.15a) while the C^1 functions B^a_b have to be solutions of (A.13). The integrability conditions $(\partial^2/\partial s^\beta \partial s^\alpha - \partial^2/\partial s^\alpha \partial s^\beta) B^a_b(s) = 0$ for (A.13) can be written as[17]

[17] At this point one should require Δ^h to be of class C^2 which is possible if the manifolds E and M are of class C^3.

$$\left(-\frac{\partial^2 \Gamma^c_\alpha}{\partial u^\beta \partial u^b} + \frac{\partial^2 \Gamma^c_\beta}{\partial u^\alpha \partial u^b} + \frac{\partial \Gamma^d_\alpha}{\partial u^b}\frac{\partial \Gamma^c_\beta}{\partial u^d} - \frac{\partial \Gamma^d_\beta}{\partial u^b}\frac{\partial \Gamma^c_\alpha}{\partial u^d} \right)\Bigg|_{\beta(s)} B^a_c(s)$$
$$+ \frac{\partial D^a_{b\alpha}(s)}{\partial s^\beta} - \frac{\partial D^a_{b\beta}(s)}{\partial s^\alpha} = 0,$$

which conditions split into

$$0 = \frac{\partial D^a_{b\alpha}(s)}{\partial s^\beta} - \frac{\partial D^a_{b\beta}(s)}{\partial s^\alpha}, \tag{A.15b}$$

$$\begin{aligned} 0 &= \left(-\frac{\partial^2 \Gamma^c_\alpha}{\partial u^\beta \partial u^b} + \frac{\partial^2 \Gamma^c_\beta}{\partial u^\alpha \partial u^b} + \frac{\partial \Gamma^d_\alpha}{\partial u^b}\frac{\partial \Gamma^c_\beta}{\partial u^d} - \frac{\partial \Gamma^d_\beta}{\partial u^b}\frac{\partial \Gamma^c_\alpha}{\partial u^d} \right)\Bigg|_{\beta(s)} \\ &= \left(-\Gamma^d_\alpha \frac{\partial^2 \Gamma^c_\beta}{\partial u^b \partial u^d} + \Gamma^d_\beta \frac{\partial^2 \Gamma^c_\alpha}{\partial u^b \partial u^d} \right)\Bigg|_{\beta(s)}, \end{aligned} \tag{A.16}$$

where (A.14) and (5.8) were applied in the derivation of the second equality in (A.16).

Since the system of Eqs. (A.15) always has solutions, e.g. $D^b_{b\alpha}(s) = 0$, we can assert that (A.14) and (A.16) are the integrability conditions for (A.4) and, if (A.14) and (A.16) hold, every solution of (A.13), with $D^a_{b\alpha}$ satisfying (A.15), is a solution of (A.4) and vice versa.

At the end, the only unproved assertion is that $D^a_{b\alpha}$ in (A.13) equals to $\partial_\alpha(D^a_b)$ with D^a_b satisfying (A.12). Indeed, since J^k_1 is an open set and hence is contractible one, the Poincaré's lemma (see [28, Section 6.3]or [29, pp. 21, 106]) implies the existence of functions D^a_b on J^k_1 such that $D^b_{b\alpha}(s) = \partial_\alpha(D^a_b)(s)$, due to (A.15b); inserting this result into (A.15a), we get (A.12). □

Remark A.3. Regardless that the conditions (A.10b) look quite special, they are identically valid for connections with

$$\Gamma^a_\alpha = -(\Gamma^a_{b\alpha} \circ \pi) \cdot u^b + G^a_\alpha \circ \pi, \tag{A.17}$$

where $\Gamma^a_{b\alpha}$ and G^a_α are C^2 functions on $\pi(\beta(J^k))$. In particular, of this kind are the affine and linear connections on vector bundles—see Propositions 3.1 and 3.2.

At last, we shall formulate the main result of the above considerations as a combination of Proposition A.1 and Lemma A.2.

Theorem A.1. *Let (E, π, M) be a C^3 bundle endowed with a C^2 connection. Under the hypotheses made and notation introduced before Proposition* A.1, *there exist solutions of the normal frame/coordinates Eq.* (5.1) *if and only if the connection's coefficients satisfy Eqs.* (A.10). *If these equations hold, all coordinates normal on $\beta(J^k_1)$ are given on U_1 by* (A.5), *where B^a_b are solutions of* (A.11), *with D^a_b being solutions of* (A.12).

Remark A.4. If there are $s_0 \in J^k$ and $\alpha \in \{1, \ldots, k\}$ such that the vector $\dot\beta_\alpha(s_0)$ is a vertical vector, $\dot\beta_\alpha(s_0) \in \Delta^v_{\beta(s_0)}$, then Theorem A.1 remains true with the only correction that the coordinates $\{u^I\}$ will *not be bundle* coordinates. If this is the case, the constructed coordinates $\{\tilde{u}^I\}$ will be solutions of (5.1), but we cannot assert that they are bundle coordinates which are (locally) normal along β in a neighborhood of the point $\beta(s_0)$.

Theorem A.1 provides a necessary and sufficient condition for the existence of local coordinates in a neighborhood of $\beta(s_0)$ for any $s_0 \in J^k$ which are locally normal along β, i.e. on $\beta(J_1^k)$ for some open subset $J_1^k \subseteq J^k$ containing s_0. Moreover, if this condition is valid, the theorem describes locally all coordinates normal along β.

Theorem A.1 can be generalized by requiring β to be locally injective instead of injective, i.e. for each $s \in J$ to exist subset $J_s^k \subseteq J^k$ such that $J_s^k \ni s$ and the restricted mapping $\beta|_{J_s^k}$ to be injective. Besides, if one needs a version of the above results for complex bundles, they should be considered as real ones (with doubled dimension of the manifolds) for which are applicable the above considerations.

References

[1] B. Riemann, Über die Hypothesen welche de Geometrie zugrunde liegen (On the hypotheses underlying the geometry), Götingen Abh. 13 (1868) 1–20 (in German);
B. Riemann, Habilitationsschrift, Ph.D. Thesis, 1854 (in German).

[2] O. Veblen, Normal coordinates for the geometry of paths, Proc. Natl. Acad. 8 (1922) 192–197.

[3] E. Fermi, Sopra i fenomeni che avvengono in vicinonza di una linear oraria (On phenomena near a world line), Atti R. Accad Lincei Rend., Cl. Sci. Fis. Mat. Nat. 31 (1) (1922) 21–23, 51–52 (Russian translation in [30, pp. 64–71]) (in Italian).

[4] T. Levi-Civita, Sur l'écart géodésique, Math. Ann. 97 (1926) 291–320.

[5] Luther P. Eisenhart, Non-Riemannian Geometry, vol. VIII of Colloquium Publications, American Mathematical Society New York, 1927.

[6] L.Ó. Raifeartaigh, Fermi coordinates, Proc. R. Irish Acad. 59 Sec. A (2) (1998) 15–24.

[7] Bozhidar Z. Iliev, Normal frames and the validity of the equivalence principle. I. Cases in a neighborhood and at a point, J. Phys. A: Math. Gen. 29 (21) (1996) 6895–6901. http://www.arXiv.orge-Print archive, E-print No. gr-qc/9608019, August 1998.

[8] Bozhidar Z. Iliev, Normal frames and the validity of the equivalence principle. II. The case along paths, J. Phys. A: Math. Gen. 30 (12) (1997) 4327–4336. http://www.arXiv.orge-Print archive, E-print No. gr-qc/9709053, September 1997.

[9] Bozhidar Z. Iliev, Normal frames and the validity of the equivalence principle. III. The case along smooth maps with separable points of self-intersection, J. Phys. A: Math. Gen. 31 (4) (1998) 1287–1296. http://www.arXiv.orge-Print archive, E-print No. gr-qc/9805088, May 1998.

[10] Bozhidar Z. Iliev, Normal frames for derivations and linear connections and the equivalence principle, J. Geom. Phys. 45 (1/2) (2003) 24–53. http://www.arXiv.orge-Print archive, E-print No. hep-th/0110194, October 2001.

[11] Bozhidar Z. Iliev, Normal frames and linear transports along paths in vector bundles. http://www.arXiv.orge-Print archive, E-print No. gr-qc/9809084, September 1998. Revised March 2005.

[12] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. I, Interscience Publishers, New York, 1963 (Russian translation: Nauka, Moscow, 1981);
S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. II, Interscience Publishers, New York, 1969 (Russian translation: Nauka, Moscow, 1981).

[13] F.W. Warner, Foundations of Differentiable Manifolds and Lie Groups, Springer-Verlag, New York, 1983 (Russian translation: Mir, Moscow, 1987).

[14] Walter A. Poor, Differential Geometric Structures, McGraw-Hill Book Company Inc., New York, 1981.

[15] M. Spivak, A Comprehensive Introduction to Differential Geometry, vol. 1, Publish or Perish, Boston, 1970.

[16] L. Mangiarotti, G. Sardanashvily, Connections in Classical and Quantum Field Theory, World Scientific, Singapore, 2000.
[17] Bernard F. Schutz, Geometrical Methods of Mathematical Physics, Cambridge University Press, Cambridge, 1982 (Russian translation: Mir, Moscow, 1984).
[18] D.J. Saunders, The Geometry of Jet Bundles, Cambridge University Press, Cambridge, 1989.
[19] Maido Rahula, New Problems in Differential Geometry, vol. 8 of Series on Soviet and East European Mathematics, World Scientific, Singapore, 1993 172 pp.
[20] Ph. Hartman, Ordinary Differential Equations, John Wiley & Sons, New York, 1964.
[21] S. Kobayashi, K. Nomizu, Foundations of Differential Geometry, vol. I, Interscience Publishers, New York, 1963 (Russian translation: Nauka, Moscow, 1981).
[22] Laurent Schwartz, Analyse mathématique, vol. I, Hermann, Paris, 1967 (Russian translation: Mir, Moscow, 1972) (in French).
[23] J. Dieudonné, Foundations of Modern Analysis, Academic Press, New York, 1960.
[24] Bozhidar Z. Iliev, Is the principle of equivalence a principle?, J. Geom. Phys., 24 (3) (1998) 209–222. http://www.arXiv.orge-Print archive, E-print No. gr-qc/9806062, June 1998.
[25] Bozhidar Z. Iliev, Equivalence principle in classical electrodynamics, in: Andrew Chubykalo, Vladimir Onoochin, Augusto Espinoza, Roman Smirnov-Rueda (Eds.), Has the Last Word Been Said on Classical Electrodynamics?—New Horizons, Rinton Press, Princeton, NJ, 2004, pp. 385–403. http://www.arXiv.orge-Print archive, E-print No. gr-qc/0303002, March 2003.
[26] R. Bellman, Introduction to Matrix Analysis, McGraw-Hill Book Company Inc., New York, 1960 (Russian translation: Nauka, Moscow, 1978).
[27] F.R. Gantmacher, The Theory of Matrices, vol. 1, Chelsea Pub. Co., New York, NY, 1960 (Translation from Russian. The right English transliteration of the author's name is Gantmakher).
[28] C. Nash, S. Sen, Topology and Geometry for physicists, Academic Press, London, 1983.
[29] M. Göckeler, T. Schücker, Differential Geometry, Gauge Theories, and Gravity, Cambridge University Press, Cambridge, 1987.
[30] E. Fermi, Scientific Papers, 1921–1938, Italy, vol. I, Nauka, Moscow, 1971 (in Russian).